MAS651 Theory of Stochastic Processes
Homework #9

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$.

We also assume throughout this assignment that the underlying probability space is the canonical probability space $(C_d, \mathcal{C}_d, \mathbb{P}_{\mathbf{x}})$ we have constructed in *Section 7.1* in [1]. Here, $C_d := C\left([0,+\infty), \mathbb{R}^d\right)$ refers to the function space of all continuous functions from $\mathbb{R}_+ = [0,+\infty)$ to $\mathbb{R}^d$, $\mathcal{C}_d$ denotes the σ-field on C_d generated by the coordinate maps, *i.e.*,

$$\mathcal{C}_d = \sigma\left(\left\{\{\omega(\cdot) \in C_d : \omega(t_1) \in A_1, \cdots, \omega(t_n) \in A_n\} : 0 \leq t_1 < \cdots < t_n < +\infty, A_1, \cdots, A_n \in \mathcal{R}^d\right\}\right),$$

where $\mathcal{R}^d := \mathcal{B}\left(\mathbb{R}^d\right)$ is the Borel σ-field on $\mathbb{R}^d$, and $\mathbb{P}_{\mathbf{x}}$ is the *canonical probability measure* on $(C_d, \mathcal{C}_d)$ so that the continuous-time stochastic process $\{B(t) : t \in \mathbb{R}_+\}$ consists of coordinate maps on $(C_d, \mathcal{C}_d)$ forms a d-dimensional Brownian motion such that $\mathbb{P}_{\mathbf{x}}(\{\omega(\cdot) \in C_d : \omega(0) = \mathbf{x}\}) = 1$. In other words, $\mathbf{x} \in \mathbb{R}^d$ indicates the starting point of $\{B(t) : t \in \mathbb{R}_+\}$ and this d-dimensional Brownian motion is often called the *canonical d-dimensional Brownian motion* or the *Wiener process* [3]. The probability measure $\mathbb{P}_{\mathbf{x}}$ is often called the *Wiener measure* with initial state $\mathbf{x} \in \mathbb{R}^d$, and the canonical probability space $(C_d, \mathcal{C}_d, \mathbb{P}_{\mathbf{x}})$ is referred to as the *Wiener probability space* [2].

Problem 1 (*Exercise 7.4.1.* in [1])**.**

(i) We first prove that the continuous-time stochastic process $\left\{B^1(T_a) : a \in \mathbb{R}_+\right\}$ has independent and stationary increments. Since

$$T_a = \inf\left\{t \in \mathbb{R}_+ : B^2(t) = a\right\} = \inf\left\{t \in \mathbb{R}_+ : B(t) \in \mathbb{R} \times [a,+\infty)\right\}$$

when $B(0) = (0,0) \in \mathbb{R}^2$, T_a is a stopping time with respect to the right-continuous filtration $\{\mathcal{F}(t) : t \in \mathbb{R}_+\}$ defined on $(C_2, \mathcal{C}_2)$ generated by the canonical two-dimensional Brownian motion $\{B(t) : t \in \mathbb{R}_+\}$, *i.e.*,

$$\mathcal{F}(t) := \bigcap_{\mathbf{x} \in \mathbb{R}^2} \sigma\left(\mathcal{F}_t^+ \cup \mathcal{N}_{\mathbb{P}_{\mathbf{x}}}\right),$$

where $\mathcal{N}_{\mathbb{P}_{\mathbf{x}}} := \{A \subseteq C_2 : A \subseteq B \text{ for some } B \in \mathcal{C}_2 \text{ such that } \mathbb{P}_{\mathbf{x}}\{B\} = 0\}$ denotes the collection of all $\mathbb{P}_{\mathbf{x}}$-null sets. Now, we choose any $0 < a_1 < a_2 < \cdots < a_n < +\infty$ and $f_1, f_2, \cdots, f_n \in \mathcal{B}(\mathbb{R}, \mathcal{R})$, where $\mathcal{B}(\mathbb{S}, \mathcal{S})$ refers to the collection of all bounded measurable functions from $(\mathbb{S},\mathcal{S})$ into $(\mathbb{R},\mathcal{R})$. Due to *Theorem 7.3.8* in [1], we know that $B^1(T_a)$ is $\mathcal{F}(T_a)$-measurable for every $a\in\mathbb{R}_+$. Thanks to the continuity of Brownian paths, we have $T_{a_1}\le T_{a_2}\le\cdots\le T_{a_n}$ when $B(0)=(0,0)$. Thus by *Theorem 7.3.6* in [1],

$$\mathcal{F}(T_{a_1})\subseteq\mathcal{F}(T_{a_2})\subseteq\cdots\subseteq\mathcal{F}(T_{a_n}),$$

thereby $X_k:=B^1(T_{a_k})-B^1(T_{a_{k-1}})$ is $\mathcal{F}(T_{a_k})$-measurable for every $k\in[n]$, where $a_0:=0$.

Claim 1. *For any $a\in\mathbb{R}_+$, we have $\mathbb{P}_{(0,0)}\left\{T_a<+\infty\right\}=1$.*

Proof of Claim 1.

To begin with, we observe that if $\left\{B(t)=\left(B^1(t),B^2(t),\cdots,B^d(t)\right):t\in\mathbb{R}_+\right\}$ is a standard d-dimensional Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, then $\left\{B^i(t):t\in\mathbb{R}_+\right\}$, $i\in[d]$, are independent standard one-dimensional Brownian motions defined on $(\Omega,\mathcal{F},\mathbb{P})$. Therefore, $\left\{B^2(t):t\in\mathbb{R}_+\right\}$ is a standard one-dimensional Brownian motion under $\mathbb{P}_{(0,0)}$. So by *Theorem 7.2.8* in [1], we arrive at

$$\mathbb{P}_{(0,0)}\left\{\limsup_{t\to+\infty}\frac{B^2(t)}{\sqrt{t}}=+\infty\right\}=1.$$

From the relation $\left\{\limsup_{t\to+\infty}\frac{B^2(t)}{\sqrt{t}}=+\infty\right\}\subseteq\{T_a<+\infty\}$, we obtain $\mathbb{P}_{(0,0)}\left\{T_a<+\infty\right\}=1$ as desired.

□

Claim 2. *For any $0\le u<v<+\infty$ and $\varphi\in\mathcal{B}(\mathbb{R},\mathcal{R})$, one has*

$$\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_v)-B^1(T_u)\right\}\middle|\,\mathcal{F}(T_u)\right]\cdot\mathbb{1}_{\{T_u<+\infty\}}\overset{\mathbb{P}_{(0,0)}\text{-a.s.}}{=}\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_{v-u})\right\}\right]\cdot\mathbb{1}_{\{T_u<+\infty\}}. \tag{1}$$

Proof of Claim 1.

For any $\omega(\cdot)\in C_2$, we have

$$\begin{aligned}(T_v\circ\theta_{T_u})(\omega)&=\inf\left\{t\in\mathbb{R}_+:B_{t+T_u}(\omega)=v\right\}\\&\overset{\text{(a)}}{=}\inf\left\{t\in\mathbb{R}_+:B_t(\omega)=v\right\}-T_u(\omega)\\&=T_v(\omega)-T_u(\omega),\end{aligned}\tag{2}$$

when $B(0)=(0,0)$, where the step (a) holds since $T_u(\omega)<T_v(\omega)$, which comes from the continuity of $\omega(\cdot)$. Therefore, we see that

$$\begin{aligned}B^1_{T_v(\omega)}(\omega)-B^1_{T_u(\omega)}(\omega)&=B^1_{T_v(\omega)-T_u(\omega)}\left(\theta_{T_u}(\omega)\right)-B^1_0\left(\theta_{T_u}(\omega)\right)\\&\overset{\text{(b)}}{=}B^1_{(T_v\circ\theta_{T_u})(\omega)}\left(\theta_{T_u}(\omega)\right)-B^1_{(0\circ\theta_{T_u})(\omega)}\left(\theta_{T_u}(\omega)\right)\\&=\left\{\left(B^1_{T_v}-B^1_0\right)\circ\theta_{T_u}\right\}(\omega),\end{aligned}\tag{3}$$

where the step (b) makes use of the equation (2). In short, we obtain

$$B^1(T_v)-B^1(T_u)=\left(B^1(T_v)-B^1(0)\right)\circ\theta_{T_u}\tag{4}$$

on C_2. So we arrive at

$$\begin{aligned}\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_v)-B^1(T_u)\right\}\middle|\,\mathcal{F}(T_u)\right]\cdot\mathbb{1}_{\{T_u<+\infty\}}&\overset{\text{(c)}}{=}\mathbb{E}_{(0,0)}\left[\left\{\varphi\circ\left(B^1(T_v)-B^1(0)\right)\right\}\circ\theta_{T_u}\middle|\,\mathcal{F}(T_u)\right]\cdot\mathbb{1}_{\{T_u<+\infty\}}\\&\overset{\text{(d)}}{=}\mathbb{E}_{(B^1(T_u),u)}\left[\varphi\left\{B^1(T_v)-B^1(0)\right\}\right]\cdot\mathbb{1}_{\{T_u<+\infty\}}\\&\overset{\text{(e)}}{=}\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_{v-u})\right\}\right]\cdot\mathbb{1}_{\{T_u<+\infty\}},\end{aligned}$$

where the above steps (c)–(e) can be justified as follows:

(c) the equation (4);

(d) the strong Markov property of Brownian motions (*Theorem 7.3.9* in [1]) together with the fact that $B^2(T_u) = u$ when $T_u < +\infty$;

(e) translation invariance of Brownian motions.

□

So we may deduce that for any $0 \le u < v < +\infty$ and $\varphi \in \mathcal{B}(\mathbb{R}, \mathcal{R})$,

$$\begin{aligned}
\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_v) - B^1(T_u)\right\}\right] &\overset{\text{(f)}}{=} \mathbb{E}_{(0,0)}\left[\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_v) - B^1(T_u)\right\} \cdot \mathbb{1}_{\{T_u<+\infty\}} \middle| \mathcal{F}(T_u)\right]\right] \\
&\overset{\text{(g)}}{=} \mathbb{E}_{(0,0)}\left[\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_v) - B^1(T_u)\right\} \middle| \mathcal{F}(T_u)\right] \cdot \mathbb{1}_{\{T_u<+\infty\}}\right] \\
&\overset{\text{(h)}}{=} \mathbb{E}_{(0,0)}\left[\mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_{v-u})\right\}\right] \cdot \mathbb{1}_{\{T_u<+\infty\}}\right] \\
&\overset{\text{(i)}}{=} \mathbb{E}_{(0,0)}\left[\varphi\left\{B^1(T_{v-u})\right\}\right],
\end{aligned} \tag{5}$$

where the above steps (f)–(i) can be validated as follows:

(f) Claim 1;

(g) $\{T_u < +\infty\} \in \mathcal{F}(T_u)$;

(h) Claim 2;

(i) Claim 1.

Therefore, the continuous-time stochastic process $\left\{B^1(T_a) : a \in \mathbb{R}_+\right\}$ has stationary increments under $\mathbb{P}_{(0,0)}$.

On the other hand, one can see that

$$\begin{aligned}
\mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n} f_k(X_k)\right] &= \mathbb{E}_{(0,0)}\left[\mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n} f_k(X_k) \middle| \mathcal{F}(T_{a_{n-1}})\right]\right] \\
&\overset{\text{(j)}}{=} \mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n-1} f_k(X_k) \cdot \mathbb{E}_{(0,0)}\left[f_n(X_n) \middle| \mathcal{F}(T_{a_{n-1}})\right]\right] \\
&\overset{\text{(k)}}{=} \mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n-1} f_k(X_k) \cdot \mathbb{E}_{(0,0)}\left[f_n\left\{B^1(T_{a_n}) - B^1(T_{a_{n-1}})\right\} \middle| \mathcal{F}(T_{a_{n-1}})\right] \cdot \mathbb{1}_{\{T_{a_{n-1}}<+\infty\}}\right] \\
&\overset{\text{(l)}}{=} \mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n-1} f_k(X_k) \cdot \mathbb{E}_{(0,0)}\left[f_n\left\{B^1(T_{a_n-a_{n-1}})\right\}\right] \cdot \mathbb{1}_{\{T_{a_{n-1}}<+\infty\}}\right] \\
&\overset{\text{(m)}}{=} \mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n-1} f_k(X_k)\right] \mathbb{E}_{(0,0)}\left[f_n\left\{B^1(T_{a_n-a_{n-1}})\right\}\right] \\
&\overset{\text{(n)}}{=} \mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n-1} f_k(X_k)\right] \mathbb{E}_{(0,0)}\left[f_n(X_n)\right],
\end{aligned} \tag{6}$$

where the above steps (j)–(n) hold due to the following reasons:

(j) $f_k(X_k)$ is $\mathcal{F}(T_{a_{n-1}})$-measurable for every $k \in [n-1]$;

(k) Claim 1;

(l) Claim 2;

(m) Claim 1;

(n) the equation (5).

By employing the equation (6) repeatedly, we eventually obtain

$$\mathbb{E}_{(0,0)}\left[\prod_{k=1}^{n} f_k(X_k)\right] = \prod_{k=1}^{n} \mathbb{E}_{(0,0)}\left[f_k(X_k)\right]$$

for every $f_1, f_2, \cdots, f_n \in \mathcal{B}(\mathbb{R}, \mathcal{R})$. Hence, $X_1, X_2, \cdots, X_n$ are independent and this implies that $\left\{B^1(T_a) : a \in \mathbb{R}_+\right\}$ has independent increments under $\mathbb{P}_{(0,0)}$.

(ii) Given any $a \in (0, +\infty)$, consider the $\left(\mathbb{R}^2, \mathcal{R}^2\right)$-valued continuous-time stochastic process $\{X(t) : t \in \mathbb{R}_+\}$ defined on $(C_2, \mathcal{C}_2)$ by

$$X_t(\omega) = \left(X_t^1(\omega), X_t^2(\omega)\right) := \frac{1}{a} B_{a^2 t}(\omega),\ t \in \mathbb{R}_+.$$

By the scaling invariance of Brownian motions, $\{X(t) : t \in \mathbb{R}_+\}$ is also a standard two-dimensional Brownian motion under $\mathbb{P}_{(0,0)}$ and thus it has the same joint law as the canonical standard two-dimensional Brownian motion $\{B(t) : t \in \mathbb{R}_+\}$ under $\mathbb{P}_{(0,0)}$. We can observe that

$$\begin{aligned} B^1(T_a) &= B^1\left(\inf\left\{t \in \mathbb{R}_+ : B^2(t) = a\right\}\right) \\ &= B^1\left(a^2 \inf\left\{t \in \mathbb{R}_+ : B^2(a^2 t) = a\right\}\right) \\ &= B^1\left(a^2 \inf\left\{t \in \mathbb{R}_+ : X^2(t) = 1\right\}\right) \\ &= a \cdot X^1\left(\inf\left\{t \in \mathbb{R}_+ : X^2(t) = 1\right\}\right). \end{aligned} \tag{7}$$

We define a measurable function $f : (C_2, \mathcal{C}_2) \to (\mathbb{R}, \mathcal{R})$ by

$$f(\Phi(\cdot)) = f\left(\Phi_1(\cdot), \Phi_2(\cdot)\right) := \Phi_1\left(\inf\left\{t \in \mathbb{R}_+ :\ \Phi_2(t) = 1\right\}\right) \cdot \mathbb{1}_{\{\Phi(\cdot) \in C_2 : \Phi_2(t) = 1 \text{ for some } t \in \mathbb{R}_+\}}.$$

Since two $(C_2, \mathcal{C}_2)$-valued random variables $B_\bullet : (C_2, \mathcal{C}_2) \to (C_2, \mathcal{C}_2)$ and $X_\bullet : (C_2, \mathcal{C}_2) \to (C_2, \mathcal{C}_2)$ given by

$$B_\bullet(\omega)(t) := B_t(\omega) \quad \text{and} \quad X_\bullet(\omega)(t) := X_t(\omega),\ \forall t \in \mathbb{R}_+,\ \omega(\cdot) \in C_2,$$

have the same distributions under $\mathbb{P}_{(0,0)}$, $f \circ B_\bullet : (C_2, \mathcal{C}_2) \to (\mathbb{R}, \mathcal{R})$ and $f \circ X_\bullet : (C_2, \mathcal{C}_2) \to (\mathbb{R}, \mathcal{R})$ have the same distributions under $\mathbb{P}_{(0,0)}$. Thus, we have from Claim 1 together with the fact that two $(C_2, \mathcal{C}_2)$-valued random variables $B_\bullet : (C_2, \mathcal{C}_2) \to (C_2, \mathcal{C}_2)$ and $X_\bullet : (C_2, \mathcal{C}_2) \to (C_2, \mathcal{C}_2)$ have the same law under $\mathbb{P}_{(0,0)}$ that

$$\mathbb{P}_{(0,0)}\left\{B(t) = 1 \text{ for some } t \in \mathbb{R}_+\right\} = \mathbb{P}_{(0,0)}\left\{X(t) = 1 \text{ for some } t \in \mathbb{R}_+\right\} = 1. \tag{8}$$

Hence we see from the equation (8) that

$$\begin{aligned} B^1\left(\inf\left\{t \in \mathbb{R}_+ : B^2(t) = 1\right\}\right) &\overset{\mathbb{P}_{(0,0)}\text{-a.s.}}{=} f \circ B_\bullet \\ &\overset{d}{=} f \circ X_\bullet \\ &\overset{\mathbb{P}_{(0,0)}\text{-a.s.}}{=} X^1\left(\inf\left\{t \in \mathbb{R}_+ : X^2(t) = 1\right\}\right), \end{aligned}$$

thereby

$$B^1\left(\inf\left\{t \in \mathbb{R}_+ : B^2(t) = 1\right\}\right) \stackrel{d}{=} X^1\left(\inf\left\{t \in \mathbb{R}_+ : X^2(t) = 1\right\}\right). \tag{9}$$

Taking two pieces (7) and (9) collectively yields the desired result $B^1(T_a) \stackrel{d}{=} a \cdot B^1(T_1)$ under $\mathbb{P}_{(0,0)}$.

(iii) Finally, we show that $B^1(T_a)$ follows the Cauchy distribution with parameter $(0, a) \in \mathbb{R} \times (0, +\infty)$. From the fact that $\left\{B^1(t) : t \in \mathbb{R}_+\right\}$ and $\left\{B^2(t) : t \in \mathbb{R}_+\right\}$ are independent standard one-dimensional Brownian motions under $\mathbb{P}_{(0,0)}$, we observe that $\left\{B^1(t) : t \in \mathbb{R}_+\right\}$ and $T_a = \inf\left\{t \in \mathbb{R}_+ : B^2(t) = a\right\}$ are independent under $\mathbb{P}_{(0,0)}$. Hence for any $t \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}_{(0,0)}\left\{B^1(T_a) \le t\right\} &= \mathbb{E}_{(0,0)}\left[\mathbb{P}_{(0,0)}\left\{B^1(T_a) \le t \middle| T_a\right\}\right] \\
&= \int_0^\infty f_{T_a}(s) \cdot \mathbb{P}_{(0,0)}\left\{B^1(T_a) \le t \middle| T_a = s\right\} \mathrm{d}s \\
&\stackrel{\text{(o)}}{=} \int_0^\infty \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) \cdot \mathbb{P}_{(0,0)}\left\{B^1(s) \le t\right\} \mathrm{d}s \\
&\stackrel{\text{(p)}}{=} \int_0^\infty \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) \left[\int_{-\infty}^t \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \mathrm{d}x\right] \mathrm{d}s \\
&\stackrel{\text{(q)}}{=} \int_{-\infty}^t \frac{a}{2\pi} \left[\int_0^\infty \frac{1}{s^2} \exp\left(-\frac{a^2 + x^2}{2s}\right) \mathrm{d}s\right] \mathrm{d}x \\
&\stackrel{\text{(r)}}{=} \int_{-\infty}^t \frac{a}{2\pi} \left[\int_\infty^0 u^2 \exp\left(-\frac{u\left(a^2 + x^2\right)}{2}\right) \cdot \left(-\frac{1}{u^2}\right) \mathrm{d}u\right] \mathrm{d}x \\
&= \int_{-\infty}^t \frac{1}{\pi} \cdot \frac{a}{a^2 + x^2} \mathrm{d}x,
\end{aligned}$$

where $f_{T_a}(\cdot)$ is the probability density function of the first hitting time T_a to $a \in (0, +\infty)$ which is given by

$$f_{T_a}(s) := \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) \quad \text{for } s \in (0, +\infty), \tag{10}$$

whose derivation can be found in *Equation (7.4.6)* in [1], and the steps (o)–(r) can be verified as follows:

(o) since $\left\{B^1(t) : t \in \mathbb{R}_+\right\}$ and T_a are indendent under $\mathbb{P}_{(0,0)}$, we have for every $s \in (0, +\infty)$,

$$\begin{aligned}
\mathbb{P}_{(0,0)}\left\{B^1(T_a) \le t \middle| T_a = s\right\} &= \mathbb{P}_{(0,0)}\left\{B^1(s) \le t \middle| T_a = s\right\} \\
&= \mathbb{P}_{(0,0)}\left\{B^1(s) \le t\right\}.
\end{aligned}$$

(p) since $\left\{B^1(t) : t \in \mathbb{R}_+\right\}$ is a standard one-dimensional Brownian motion, $B^1(s) \sim \mathsf{N}(0, s)$ under $\mathbb{P}_{(0,0)}$;

(q) it's possible to interchange the order of integrals due to the Fubini-Tonelli's theorem;

(r) the substitution $u = \frac{1}{s}$.

Hence, the probability density function of $B^1(T_a)$, $a \in (0, +\infty)$, is given by

$$f_{B^1(T_a)}(x) = \frac{1}{\pi} \cdot \frac{a}{a^2 + x^2}, \ \forall x \in \mathbb{R},$$

and this completes the proof of the desired result.

Problem 2 (*Exercise 7.4.2.* in [1]).

To begin with, we provide a proof of the following result:

Lemma 1 (*Exercise 7.2.1* in [1]).

Let $\tau_0 := \inf\{s \in (0,+\infty) : B(s) = 0\}$, *and* $R := \inf\{t \in (1,+\infty) : B(t) = 0\}$. *Then for any* $x \in \mathbb{R}$ *and* $t \in (0,+\infty)$, *one has*

$$\mathbb{P}_x\{R > 1+t\} = \int_{\mathbb{R}} p_1(x,y) \cdot \mathbb{P}_y\{\tau_0 > t\}\,\mathrm{d}y, \tag{11}$$

where $\{p_t(\cdot,\cdot) : t \in \mathbb{R}_+\}$ *refers to the Markov semi-group for the one-dimensional Brownian motion, i.e.,*

$$p_t(x,y) := \begin{cases} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(y-x)^2}{2t}\right\} & \text{if } t > 0; \\ \delta_x(y) & \text{if } t = 0. \end{cases}$$

Proof of Lemma 1.

We first note that for any $t \in (0,+\infty)$,

$$\begin{aligned} \{B(s) \neq 0 \text{ for all } s \in (0,t]\} &= \{\tau_0 > t\}; \\ \{B(1+s) \neq 0 \text{ for all } s \in (0,t]\} &= \{R > 1+t\} \end{aligned} \tag{12}$$

which follow from the continuity of the Brownian paths. Thus,

$$\begin{aligned} \mathbb{P}_x\{R > 1+t\} &\overset{\text{(a)}}{=} \mathbb{P}_x\{B(1+s) \neq 0 \text{ for all } s \in (0,t]\} \\ &= \mathbb{P}_x\{\omega(\cdot) \in C_1 : (\theta_1(\omega))(s) \neq 0 \text{ for all } s \in (0,t]\} \\ &= \mathbb{E}_x\left[\mathbb{1}_{\{B(s)\neq 0 \text{ for all } s\in(0,t]\}} \circ \theta_1\right] \\ &\overset{\text{(b)}}{=} \mathbb{E}_x\left[\mathbb{1}_{\{\tau_0 > t\}} \circ \theta_1\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_x\left[\mathbb{1}_{\{\tau_0>t\}} \circ \theta_1 \middle| \mathcal{F}_1^+\right]\right] \\ &\overset{\text{(c)}}{=} \mathbb{E}_x\left[\mathbb{E}_{B(1)}\left[\mathbb{1}_{\{\tau_0>t\}}\right]\right] \\ &= \int_{\mathbb{R}} p_1(x,y) \cdot \mathbb{P}_y\{\tau_0 > t\}\,\mathrm{d}y, \end{aligned}$$

where the above steps (a)–(c) can be confirmed as follows:

(a) the relation (12);

(b) the relation (12);

(c) the Markov property for Brownian motions (*Theorem 7.2.1* in [1]).

□

Given any $a \in \mathbb{R}$, let

$$\tau_a := \inf\{t > 0 : B(t) = a\} \quad \text{and} \quad T_a := \inf\{t \in \mathbb{R}_+ : B(t) = a\}.$$

It's clear that $\tau_a = T_a$ when $B(0) \in \mathbb{R} \setminus \{a\}$.

Lemma 2. $\mathbb{P}_x\{\tau_y > t\} = \mathbb{P}_0\left\{\tau_{|y-x|} > t\right\}$ *for any* $x, y \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$.

Proof of Lemma 2.

Let us consider the continuous-time stochastic process $\{X(t) : t \in \mathbb{R}_+\}$ defined on $(C_1, \mathcal{C}_1)$ by

$$X_t(\omega) := -B_t(\omega),\ \forall t \in \mathbb{R}_+,\ \omega(\cdot) \in C_1.$$

It's easy to see that $\{X(t) : t \in \mathbb{R}_+\}$ has the same joint law under $\mathbb{P}_0$ as $\{B(t) : t \in \mathbb{R}_+\}$, thereby for any $a \in \mathbb{R}$,

$$\begin{aligned} \tau_a &= \inf\{t > 0 : B(t) = a\} \\ &= \inf\{t > 0 : X(t) = -a\} \\ &\overset{d}{=} \inf\{t > 0 : B(t) = -a\} \\ &= \tau_{-a} \end{aligned} \tag{13}$$

under $\mathbb{P}_0$. Thus we can conclude that

$$\mathbb{P}_x\{\tau_y > t\} \overset{\text{(d)}}{=} \mathbb{P}_0\{\tau_{y-x} > t\} \overset{\text{(e)}}{=} \mathbb{P}_0\left\{\tau_{|y-x|} > t\right\},$$

where the step (d) follows from the translation invariance of Brownian motions, and the step (e) makes use of (13). This completes the proof of Lemma 2.

□

By employing Lemma 2 to the equation (11) with $x = 0$, we obtain

$$\begin{aligned} \mathbb{P}_0\{R > 1+t\} &= \int_{\mathbb{R}} p_1(0,y) \cdot \mathbb{P}_y\{\tau_0 > t\}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} p_1(0,y) \cdot \mathbb{P}_0\left\{\tau_{|y|} > t\right\}\mathrm{d}y \\ &= 2\int_0^{\infty} p_1(0,y) \cdot \mathbb{P}_0\{T_y > t\}\,\mathrm{d}y, \end{aligned}$$

which leads us to

$$\begin{aligned} \mathbb{P}_0\{R \le 1+t\} &= 1 - \mathbb{P}_0\{R > 1+t\} \\ &= 1 - 2\int_0^{\infty} p_1(0,y) \cdot \mathbb{P}_0\{T_y > t\}\,\mathrm{d}y \\ &= 2\int_0^{\infty} p_1(0,y)\mathrm{d}y - 2\int_0^{\infty} p_1(0,y) \cdot \mathbb{P}_0\{T_y > t\}\,\mathrm{d}y \\ &= 2\int_0^{\infty} p_1(0,y)\left(1 - \mathbb{P}_0\{T_y > t\}\right)\mathrm{d}y \\ &= 2\int_0^{\infty} p_1(0,y) \cdot \mathbb{P}_0\{T_y \le t\}\,\mathrm{d}y. \end{aligned} \tag{14}$$

Finally, we express the equation (14) by using the explicit form of the probability density function (10) of the first hitting time T_y to $y \in (0, +\infty)$:

$$\begin{aligned} \mathbb{P}_0\{R \le 1+t\} &= 2\int_0^{\infty} p_1(0,y) \cdot \mathbb{P}_0\{T_y \le t\}\,\mathrm{d}y \\ &= 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)\left[\int_0^t \frac{y}{\sqrt{2\pi x^3}} \exp\left(-\frac{y^2}{2x}\right)\mathrm{d}x\right]\mathrm{d}y \\ &\overset{\text{(f)}}{=} 2\int_0^t \frac{1}{2\pi} x^{-\frac{3}{2}}\left[\int_0^{\infty} y \exp\left\{-\frac{y^2}{2}\left(1 + \frac{1}{x}\right)\right\}\mathrm{d}y\right]\mathrm{d}x \\ &= \int_0^t \frac{1}{\pi\sqrt{x}(1+x)}\mathrm{d}x, \end{aligned} \tag{15}$$

where the step (f) is due to the Fubini-Tonelli's theorem. By differentiating both sides of the equation (15) by t, we obtain the following explicit form of the probability density function $f_R(\cdot)$ of R:

$$f_R(1+t) = \frac{1}{\pi\sqrt{t}(1+t)}, \ \forall t \in (0, +\infty) .$$

Problem 3 (*Exercise 7.4.3.* in [1]).

(i) We first fix any $a \in (0, +\infty)$, $t \in (0, +\infty)$, and $-\infty < u < v \leq a$, and then define a stopping time $S := \inf\{u \in [0, t) : B(u) = a\}$ with respect to the right-continuous filtration $\{\mathcal{F}(t) : t \in \mathbb{R}_+\}$ generated from $\{B(t) : t \in \mathbb{R}_+\}$. Now, we define two bounded measurable functions $\varphi(\cdot, \cdot) : ([0, +\infty) \times C_1, \mathcal{B}([0, +\infty)) \otimes \mathcal{C}_1) \to (\mathbb{R}, \mathcal{R})$ and $\tilde{\varphi}(\cdot, \cdot) : ([0, +\infty) \times C_1, \mathcal{B}([0, +\infty)) \otimes \mathcal{C}_1) \to (\mathbb{R}, \mathcal{R})$ by

$$\begin{aligned} \varphi(s, \omega(\cdot)) &:= \mathbb{1}_{\{s<t,\ u<\omega(t-s)<v\}}; \\ \tilde{\varphi}(s, \omega(\cdot)) &:= \mathbb{1}_{\{s<t,\ 2a-v<\omega(t-s)<2a-u\}}. \end{aligned}$$

Then it's clear that for any $s \in [0, t)$,

$$\begin{aligned} \varphi(s, \theta_s(\omega)(\cdot)) &:= \mathbb{1}_{\{s<t,\ u<\omega(t)<v\}}; \\ \tilde{\varphi}(s, \theta_s(\omega)(\cdot)) &:= \mathbb{1}_{\{s<t,\ 2a-v<\omega(t)<2a-u\}}. \end{aligned}$$

So we arrive at if $B(0) = 0$, then for every $\omega(\cdot) \in C_1$,

$$\begin{aligned} &\varphi(S(\omega), \theta_S(\omega)(\cdot)) \cdot \mathbb{1}_{\{S<+\infty\}}(\omega) := \mathbb{1}_{\{S<+\infty,\ u<B(t)<v\}}(\omega); \\ &\tilde{\varphi}(S(\omega), \theta_S(\omega)(\cdot)) \cdot \mathbb{1}_{\{S<+\infty\}}(\omega) := \mathbb{1}_{\{S<+\infty,\ 2a-v<B(t)<2a-u\}}(\omega) \overset{\text{(a)}}{=} \mathbb{1}_{\{2a-v<B(t)<2a-u\}}(\omega), \end{aligned} \tag{16}$$

where the step (a) holds since $\{2a - v < B(t) < 2a - u\} \subseteq \{S < +\infty\} = \{T_a < t\}$ if $B(0) = 0$, which follows from the continuity of the Brownian paths. Thanks to the symmetry of standard Brownian motions, we can easily observe that both $\{B(t) - a : t \in \mathbb{R}_+\}$ and $\{a - B(t) : t \in \mathbb{R}_+\}$ are standard one-dimensional Brownian motions under $\mathbb{P}_a$. So $\{B(t) : t \in \mathbb{R}_+\}$ and $\{2a - B(t) : t \in \mathbb{R}_+\}$ have the same joint law under $\mathbb{P}_a$. Consequently, we obtain

$$\begin{aligned} \mathbb{E}_a[\varphi(s, \cdot)] &= \mathbb{P}_a\{s < t,\ u < B(t-s) < v\} \\ &= \mathbb{P}_a\{s < t,\ u < 2a - B(t-s) < v\} \\ &= \mathbb{P}_a\{s < t,\ 2a - v < B(t-s) < 2a - u\} \\ &= \mathbb{E}_a[\tilde{\varphi}(s, \cdot)] \end{aligned} \tag{17}$$

for every $s \in \mathbb{R}_+$. By leveraging two pieces (16) and (17) together, we can deduce

$$\begin{aligned}
\mathbb{P}_0\left\{T_a < t,\ u < B(t) < v\right\} &= \mathbb{P}_0\left\{S < +\infty,\ u < B(t) < v\right\} \\
&= \mathbb{E}_0\left[\mathbb{1}_{\{S<+\infty,\ u<B(t)<v\}}\right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_0\left[\left\{\varphi\left(S,\cdot\right)\circ\theta_S\right\}\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(c)}}{=} \mathbb{E}_0\left[\mathbb{E}_0\left[\varphi\left(S,\cdot\right)\circ\theta_S\middle|\,\mathcal{F}(S)\right]\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(d)}}{=} \mathbb{E}_0\left[\mathbb{E}_{B(S)}\left[\varphi(S,\cdot)\right]\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(e)}}{=} \mathbb{E}_0\left[\mathbb{E}_a\left[\varphi(S,\cdot)\right]\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(f)}}{=} \mathbb{E}_0\left[\mathbb{E}_a\left[\tilde{\varphi}(S,\cdot)\right]\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(g)}}{=} \mathbb{E}_0\left[\mathbb{E}_{B(S)}\left[\tilde{\varphi}(S,\cdot)\right]\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(h)}}{=} \mathbb{E}_0\left[\mathbb{E}_0\left[\tilde{\varphi}\left(S,\cdot\right)\circ\theta_S\middle|\,\mathcal{F}(S)\right]\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(i)}}{=} \mathbb{E}_0\left[\left\{\tilde{\varphi}\left(S,\cdot\right)\circ\theta_S\right\}\cdot\mathbb{1}_{\{S<+\infty\}}\right] \\
&\overset{\text{(j)}}{=} \mathbb{E}_0\left[\mathbb{1}_{\{2a-v<B(t)<2a-u\}}\right] \\
&= \mathbb{P}_0\left\{2a-v < B(t) < 2a-u\right\},
\end{aligned}$$

where the above steps (b)–(j) can be justified as follows:

(b) the equation (16);

(c) $\{S < +\infty\} \in \mathcal{F}(S)$;

(d) the strong Markov property for Brownian motions (*Theorem 7.3.9* in [1]);

(e) if $S < +\infty$, then $B(S) = B(T_a) = a$;

(f) the equation (17);

(g) the same reason as the step (e);

(h) the same reason as the step (d);

(i) the same reason as the step (c);

(j) the same reason as the step (b).

(ii) For any $-\infty < u < v \leq a$, one has from the statement (i) that

$$\mathbb{P}_0\left\{T_a < t,\ u < B(t) < v\right\} = \int_{2a-v}^{2a-u} p_t(0,y)\mathrm{d}y = \int_u^v p_t(0,2a-y)\mathrm{d}y,$$

thereby we can see that

$$\begin{aligned}
\mathbb{P}_0\left\{T_a < t,\ B(t) < v\right\} &= \lim_{n\to\infty}\mathbb{P}_0\left\{T_a < t,\ -n < B(t) < v\right\} \\
&= \lim_{n\to\infty}\int_{-n}^{v} p_t(0,2a-y)\mathrm{d}y \qquad (18) \\
&\overset{\text{(k)}}{=} \int_{-\infty}^{v} p_t(0,2a-y)\mathrm{d}y,
\end{aligned}$$

where the step (k) follows from the Lebesgue's dominated convergence theorem. Then for any $x \in (-\infty, a)$, we obtain

$$
\begin{aligned}
\mathbb{P}_0\left\{T_a < t, B(t) \le x\right\} &= \lim_{n\to\infty} \mathbb{P}_0\left\{T_a < t, B(t) < x + \frac{1}{n}\right\} \\
&\overset{\text{(l)}}{=} \lim_{n\to\infty} \int_{-\infty}^{x+\frac{1}{n}} p_t(0, 2a-y)\mathrm{d}y \\
&\overset{\text{(m)}}{=} \int_{-\infty}^{x} p_t(0, 2a-y)\mathrm{d}y,
\end{aligned} \tag{19}
$$

where the step (l) makes use of the equation (18), and the step (m) is owing to the Lebesgue's dominated convergence theorem. As the last step, differentiating both sides of (19) yields the desired result.

(iii) Let $M(t) := \sup\left\{B(s) : s \in [0,t]\right\}$ for $t \in \mathbb{R}_+$. From the definition of $M(t)$, it's clear that

$$
\left\{T_a < t\right\} = \left\{M(t) \ge a\right\}.
$$

So from the equation (19), we obtain

$$
\mathbb{P}_0\left\{M(t) \ge a,\ B(t) \le x\right\} = \int_{-\infty}^{x} p_t(0, 2a-y)\mathrm{d}y.
$$

Thus we reach

$$
\begin{aligned}
\mathbb{P}_0\left\{M(t) > a,\ B(t) \le x\right\} &= \lim_{n\to\infty} \mathbb{P}_0\left\{M(t) \ge a + \frac{1}{n},\ B(t) \le x\right\} \\
&= \lim_{n\to\infty} \int_{-\infty}^{x} p_t\left(0, 2\left(a + \frac{1}{n}\right) - y\right)\mathrm{d}y \\
&\overset{\text{(n)}}{=} \int_{-\infty}^{x} p_t(0, 2a-y)\mathrm{d}y,
\end{aligned}
$$

where the step (n) follows from the Lebesgue's dominated convergence theorem. Consequently for any $a \in (0, +\infty)$ and $x \in (-\infty, a)$, we have

$$
\begin{aligned}
\mathbb{P}_0\left\{M(t) \le a,\ B(t) \le x\right\} &= \mathbb{P}_0\left\{B(t) \le x\right\} - \mathbb{P}_0\left\{M(t) > a,\ B(t) \le x\right\} \\
&= \int_{-\infty}^{x} \left\{p_t(0,y) - p_t(0, 2a-y)\right\}\mathrm{d}y.
\end{aligned} \tag{20}
$$

Hence, the joint probability density function of $(M(t), B(t))$, $f_{(M(t),B(t))}(\cdot,\cdot)$, can be evaluated by

$$
\begin{aligned}
f_{(M(t),B(t))}(a,x) &= \frac{\partial^2}{\partial a \partial x}\mathbb{P}_0\left\{M(t) \le a,\ B(t) \le x\right\} \\
&= \frac{\partial}{\partial a}\left\{p_t(0,x) - p_t(0, 2a-x)\right\} \\
&= \frac{2(2a-x)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(2a-x)^2}{2t}\right\}
\end{aligned}
$$

for every $a \in (0, +\infty)$ and $x \in (-\infty, a)$, as desired.

Problem 4 (*Exercise 7.5.3.* in [1]).

(i) We divide our proof in the following three cases:

(Case #1) $x \le a$: Then, we know that $\sigma = T_a$ and $T_a < T_b$ when $B(0) = x$ due to the continuity of the Brownian paths. Therefore, we have

$$\text{(LHS)} = \mathbb{E}_x \left[\exp(-\lambda\sigma) \cdot \mathbb{1}_{\{T_a < T_b\}} \right] = \text{(RHS)},$$

as desired.

(Case #2) $x \ge b$: Thanks to the continuity of the Brownian paths, we see that $\sigma = T_b$ and $T_a > T_b$ for this case. Thus,

$$\begin{aligned}
\text{(LHS)} &= \mathbb{E}_x \left[\exp\{-\lambda (T_a - T_b)\} \cdot \exp(-\lambda T_b) \cdot \mathbb{1}_{\{T_a > T_b\}} \right] \\
&\overset{\text{(a)}}{=} \mathbb{E}_x \left[\exp\{-\lambda (T_a \circ \theta_{T_b})\} \cdot \exp(-\lambda T_b) \cdot \mathbb{1}_{\{T_a > T_b\}} \right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_x \left[\mathbb{E}_x \left[\exp(-\lambda T_a) \circ \theta_{T_b} \middle| \mathcal{F}(T_b) \right] \cdot \mathbb{1}_{\{T_b < +\infty\}} \cdot \exp(-\lambda T_b) \cdot \mathbb{1}_{\{T_a > T_b\}} \right] \\
&\overset{\text{(c)}}{=} \mathbb{E}_x \left[\mathbb{E}_{B(T_b)} \left[\exp(-\lambda T_a) \right] \cdot \mathbb{1}_{\{T_b < +\infty\}} \cdot \exp(-\lambda T_b) \cdot \mathbb{1}_{\{T_a > T_b\}} \right] \\
&\overset{\text{(d)}}{=} \mathbb{E}_x \left[\mathbb{E}_b \left[\exp(-\lambda T_a) \right] \cdot \mathbb{1}_{\{T_b < +\infty\}} \cdot \exp(-\lambda\sigma) \cdot \mathbb{1}_{\{T_a > T_b\}} \right] \\
&= \mathbb{E}_x \left[\exp(-\lambda\sigma) \cdot \mathbb{1}_{\{T_a > T_b\}} \right] \cdot \mathbb{E}_b \left[\exp(-\lambda T_a) \right] \\
&\overset{\text{(e)}}{=} \text{(RHS)},
\end{aligned}$$

where the above steps (a)–(d) hold due to the following reasons:

(a) If $T_a(\omega) > T_b(\omega)$, then we see that

$$(T_a \circ \theta_{T_b})(\omega) = \inf\{t \in \mathbb{R}_+ : B_{t+T_b}(\omega) = a\} = \inf\{t \in \mathbb{R}_+ : B_t(\omega) = a\} - T_b(\omega) = T_a(\omega) - T_b(\omega)$$

for every $\omega(\cdot) \in C_1$;

(b) $\exp(-\lambda T_b)$ is $\mathcal{F}(T_b)$-measurable, $\{T_a > T_b\} \in \mathcal{F}(T_b)$ by *Exercise 7.3.5* in [1] (note that this exercise was one of the problems in Homework #8), and $\{T_a > T_b\} \subseteq \{T_b < +\infty\}$;

(c) the strong Markov property for Brownian motions (*Theorem 7.3.9* in [1]);

(d) $B(T_b) = b$ when $T_b < +\infty$, and $\sigma = T_b$, $T_a > T_b$ when $B(0) = x$;

(e) $T_a > T_b$ when $B(0) = x$.

(Case #3) $a < x < b$: From the continuity of the Brownian paths, we have $\sigma = T_a \wedge T_b$ if $B(0) = x$. Also, we may observe that if $T_a(\omega) < T_b(\omega)$, then

$$(T_b \circ \theta_{T_a})(\omega) = \inf\{t \in \mathbb{R}_+ : B_{t+T_a}(\omega) = b\} = \inf\{t \in \mathbb{R}_+ : B_t(\omega) = b\} - T_a(\omega) = T_b(\omega) - T_a(\omega), \quad (21)$$

and if $T_a(\omega) > T_b(\omega)$, then

$$(T_a \circ \theta_{T_b})(\omega) = \inf\{t \in \mathbb{R}_+ : B_{t+T_b}(\omega) = a\} = \inf\{t \in \mathbb{R}_+ : B_t(\omega) = a\} - T_b(\omega) = T_a(\omega) - T_b(\omega). \quad (22)$$

Hence, we get

$$
\begin{aligned}
\text{(LHS)} &= \mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right] \\
&= \mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\exp\left\{-\lambda\left(T_a-T_b\right)\right\}\cdot \exp\left(-\lambda T_b\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right] \\
&\overset{\text{(f)}}{=} \mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\exp\left\{-\lambda\left(T_a\circ\theta_{T_b}\right)\right\}\cdot \exp\left(-\lambda T_b\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right] \\
&\overset{\text{(g)}}{=} \mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\circ\theta_{T_b}\middle|\,\mathcal{F}(T_b)\right]\cdot \mathbb{1}_{\{T_b<+\infty\}}\cdot \exp\left(-\lambda T_b\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right] \\
&\overset{\text{(h)}}{=} \mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\mathbb{E}_{B(T_b)}\left[\exp\left(-\lambda T_a\right)\right]\cdot \mathbb{1}_{\{T_b<+\infty\}}\cdot \exp\left(-\lambda T_b\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right] \\
&\overset{\text{(i)}}{=} \mathbb{E}_x\left[\exp\left(-\lambda \sigma\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\mathbb{E}_b\left[\exp\left(-\lambda T_a\right)\right]\cdot \exp\left(-\lambda\sigma\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right] \\
&= \mathbb{E}_x\left[\exp\left(-\lambda \sigma\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\exp\left(-\lambda\sigma\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right]\cdot \mathbb{E}_b\left[\exp\left(-\lambda T_a\right)\right] \\
&= \text{(RHS)},
\end{aligned}
$$

where the steps (f)–(i) can be justified as follows:

(f) the equation (22);

(g) the same reason as the step (b);

(h) the same reason as the step (c);

(i) we have $B(T_b) = b$ if $T_b < +\infty$, and

$$
\sigma = \begin{cases} T_a & \text{if } T_a < T_b; \\ T_b & \text{if } T_a > T_b. \end{cases}
$$

Taking three cases (Case #1)–(Case #3) collectively establishes our desired result.

(ii) Here, we only consider the case $a < x < b$. From the statement (1), we obtain

$$
\begin{aligned}
\mathbb{E}_x\left[\exp\left(-\lambda\sigma\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right] + \mathbb{E}_x\left[\exp\left(-\lambda\sigma\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right]\cdot \mathbb{E}_b\left[\exp\left(-\lambda T_a\right)\right] &= \mathbb{E}_x\left[\exp\left(-\lambda T_a\right)\right] \\
&\overset{\text{(j)}}{=} \mathbb{E}_0\left[\exp\left(-\lambda T_{a-x}\right)\right] \\
&\overset{\text{(k)}}{=} \mathbb{E}_0\left[\exp\left(-\lambda T_{x-a}\right)\right] \\
&\overset{\text{(l)}}{=} \exp\left\{-(x-a)\sqrt{2\lambda}\right\},
\end{aligned}
\tag{23}
$$

where the above steps (j)–(l) can be verified as follows:

(j) the translation invariance of Brownian motions;

(k) in the proof of Lemma 2, we have seen that $T_a \overset{d}{=} T_{-a}$ under $\mathbb{P}_0$ for every $a \in \mathbb{R}$;

(l) *Theorem 7.5.7* in [1].

On the other hand, by employing exactly the same argument as the proof of (i), one can show that

$$
\begin{aligned}
\mathbb{E}_x\left[\exp\left(-\lambda\sigma\right)\cdot \mathbb{1}_{\{T_a>T_b\}}\right] + \mathbb{E}_x\left[\exp\left(-\lambda\sigma\right)\cdot \mathbb{1}_{\{T_a<T_b\}}\right]\cdot \mathbb{E}_a\left[\exp\left(-\lambda T_b\right)\right] &= \mathbb{E}_x\left[\exp\left(-\lambda T_b\right)\right] \\
&\overset{\text{(m)}}{=} \mathbb{E}_0\left[\exp\left(-\lambda T_{b-x}\right)\right] \\
&\overset{\text{(n)}}{=} \exp\left\{-(b-x)\sqrt{2\lambda}\right\},
\end{aligned}
\tag{24}
$$

where the step (m) follows from the translation invariance of Brownian motions, and the step (n) makes use of *Theorem 7.5.7* in [1]. Furthermore, from the translation invariance of Brownian motions together with the fact that $T_a \stackrel{d}{=} T_{-a}$ under $\mathbb{P}_0$ for every $a \in \mathbb{R}$, one has

$$\mathbb{E}_b\left[\exp\left(-\lambda T_a\right)\right] = \mathbb{E}_a\left[\exp\left(-\lambda T_b\right)\right] \stackrel{(o)}{=} \exp\left\{-(b-a)\sqrt{2\lambda}\right\} \tag{25}$$

for every $\lambda \in \mathbb{R}_+$, where the step (o) follows by *Theorem 7.5.7* in [1]. For the sake of readers' convenience, let us adopt the following convention: for any $\lambda \in \mathbb{R}_+$,

$$A(\lambda) := \mathbb{E}_x\left[\exp\left(-\lambda\sigma\right) \cdot \mathbb{1}_{\{T_a < T_b\}}\right] \quad \text{and} \quad B(\lambda) := \mathbb{E}_x\left[\exp\left(-\lambda\sigma\right) \cdot \mathbb{1}_{\{T_a > T_b\}}\right].$$

By combining three pieces (23)–(25) together, we eventually arrive at the following system of linear equations:

$$\begin{cases} A(\lambda) + \exp\left\{-(b-a)\sqrt{2\lambda}\right\} \cdot B(\lambda) & = \exp\left\{-(x-a)\sqrt{2\lambda}\right\}; \\ \exp\left\{-(b-a)\sqrt{2\lambda}\right\} \cdot A(\lambda) + B(\lambda) & = \exp\left\{-(b-x)\sqrt{2\lambda}\right\}. \end{cases} \tag{26}$$

By solving the linear system (26), we can obtain the following explicit forms of $A(\lambda)$ and $B(\lambda)$:

$$A(\lambda) = \frac{\sinh\left\{(b-x)\sqrt{2\lambda}\right\}}{\sinh\left\{(b-a)\sqrt{2\lambda}\right\}} \quad \text{and} \quad B(\lambda) = \frac{\sinh\left\{(x-a)\sqrt{2\lambda}\right\}}{\sinh\left\{(b-a)\sqrt{2\lambda}\right\}}$$

for every $\lambda \in \mathbb{R}_+$, as desired. This completes the proof of the statement (ii).

Problem 5 (*Exercise 7.5.4.* in [1]).

To begin with, we define $u(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$u(x,t) := x^4 - 6x^2 t + 3t^2.$$

Then, it is easy to see that $u(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ obeys the following partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \cdot \frac{\partial^2 u}{\partial x^2} = 0 \text{ on } \mathbb{R}^2.$$

So by *Theorem 7.5.8* in [1], the continuous-time stochastic process $\left\{u(t, B(t)) = B(t)^4 - 6tB(t)^2 + 3t^2 : t \in \mathbb{R}_+\right\}$ is a continuous-time martingale with respect to the right-continuous filtration $\{\mathcal{F}(t) : t \in \mathbb{R}_+\}$ generated by the canonical standard one-dimensional Brownian motion $\{B(t) : t \in \mathbb{R}_+\}$ under $\mathbb{P}_0$. Since $T \wedge \lambda$ is a bounded stopping time with respect to $\{\mathcal{F}(t) : t \in \mathbb{R}_+\}$ for any non-random constant $\lambda \in \mathbb{R}_+$, and $\{u(t, B(t)) : t \in \mathbb{R}_+\}$ is a right-continuous martingale with respect to $\{\mathcal{F}(t) : t \in \mathbb{R}_+\}$, *i.e.*, a martingale whose sample paths are all right-continuous everywhere, we may apply *Theorem 7.5.1* in [1]: for every $\lambda \in \mathbb{R}_+$,

$$0 = \mathbb{E}_0\left[u\left(T \wedge \lambda, B(T \wedge \lambda)\right)\right] = \mathbb{E}_0\left[B(T \wedge \lambda)^4\right] + 3 \cdot \mathbb{E}_0\left[(T \wedge \lambda)^2\right] - 6 \cdot \mathbb{E}_0\left[(T \wedge \lambda) \cdot B(T \wedge \lambda)^2\right]. \tag{27}$$

At this point, we can make the following observations:

(1) $T = T_a \wedge T_b$ when $B(0) = 0$ due to the continuity of the Brownian paths. By *Theorem 7.2.8* in [1], we have $\mathbb{P}_0\left\{T_a < +\infty\right\} = \mathbb{P}_0\left\{T_b < +\infty\right\} = 1$, thereby $T < +\infty$ $\mathbb{P}_0$-almost surely;

(2) $B(T \wedge \lambda) \in [a,b]$ for every $\lambda \in \mathbb{R}_+$, when $B(0) = 0$ owing to the continuity of the Brownian paths;

(3) From *Theorem 7.5.5* in [1], we know that $\mathbb{E}_0[T] = -ab < +\infty$ and this implies that $T \in L^1(C_1, \mathcal{C}_1, \mathbb{P}_0)$. Thus, one can see that when $B(0) = 0$,

$$\left|(T \wedge \lambda) \cdot B(T \wedge \lambda)^2\right| \leq \left(a^2 \vee b^2\right) \cdot T,$$

which asserts that we may apply the dominated convergence theorem to $\left\{(T \wedge \lambda) \cdot B(T \wedge \lambda)^2 : \lambda \in \mathbb{R}_+\right\}$.

From the above observations, we can conclude that

$$\mathbb{E}_0\left[B(T \wedge \lambda)^4\right] \overset{\lambda \to +\infty}{\longrightarrow} \mathbb{E}_0\left[B(T)^4\right] \ (\because \text{the bounded convergence theorem together with (1)})\,;$$
$$\mathbb{E}_0\left[(T \wedge \lambda)^2\right] \overset{\lambda \to +\infty}{\longrightarrow} \mathbb{E}_0\left[T^2\right] \ (\because \text{the monotone convergence theorem})\,;$$
$$\mathbb{E}_0\left[(T \wedge \lambda)\, B(T \wedge \lambda)^2\right] \overset{\lambda \to +\infty}{\longrightarrow} \mathbb{E}_0\left[T \cdot B(T)^2\right] \ (\because \text{the dominated convergence theorem together with (2) and (3)})$$

As the final step, we let $\lambda \to +\infty$ in the equation (27):

$$\begin{aligned} 0 &= \mathbb{E}_0\left[B(T \wedge \lambda)^4\right] + 3 \cdot \mathbb{E}_0\left[(T \wedge \lambda)^2\right] - 6 \cdot \mathbb{E}_0\left[(T \wedge \lambda) \cdot B(T \wedge \lambda)^2\right] \\ &\overset{\lambda \to +\infty}{\longrightarrow} \mathbb{E}_0\left[B(T)^4\right] + 3 \cdot \mathbb{E}_0\left[T^2\right] - 6 \cdot \mathbb{E}_0\left[T \cdot B(T)^2\right]. \end{aligned}$$

So we reach

$$\mathbb{E}_0\left[B(T)^4\right] + 3 \cdot \mathbb{E}_0\left[T^2\right] = 6 \cdot \mathbb{E}_0\left[T \cdot B(T)^2\right] \overset{\text{(a)}}{\leq} 6\left\{\mathbb{E}_0\left[T^2\right] \cdot \mathbb{E}_0\left[B(T)^4\right]\right\}^{\frac{1}{2}}, \tag{28}$$

where the step (a) follows by the Cauchy-Schwarz inequality. Our desired results, (i) $\mathbb{E}_0\left[T^2\right] \leq 4 \cdot \mathbb{E}_0\left[B(T)^4\right]$, and (ii) $\mathbb{E}_0\left[B(T)^4\right] \leq 36 \cdot \mathbb{E}_0\left[T^2\right]$, can be easily derived from the bound (28): one can see that the series of inequalities

$$3 \cdot \mathbb{E}_0\left[T^2\right] \leq \mathbb{E}_0\left[B(T)^4\right] + 3 \cdot \mathbb{E}_0\left[T^2\right] \leq 6\left\{\mathbb{E}_0\left[T^2\right] \cdot \mathbb{E}_0\left[B(T)^4\right]\right\}^{\frac{1}{2}}$$

gives the result (i), while the series of inequalities

$$\mathbb{E}_0\left[B(T)^4\right] \leq \mathbb{E}_0\left[B(T)^4\right] + 3 \cdot \mathbb{E}_0\left[T^2\right] \leq 6\left\{\mathbb{E}_0\left[T^2\right] \cdot \mathbb{E}_0\left[B(T)^4\right]\right\}^{\frac{1}{2}}$$

yields the result (ii). This completes the proof of the given statements of Problem 5.

References

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