MAS651 Theory of Stochastic Processes
Homework #8

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$.

We also assume throughout this assignment that the underlying probability space is the canonical probability space $(C_d, \mathcal{C}_d, \mathbb{P}_{\mathbf{x}})$ we have constructed in *Section 7.1* in [1]. Here, $C_d := C\left([0,+\infty), \mathbb{R}^d\right)$ refers to the function space of all continuous functions from $\mathbb{R}_+ = [0,+\infty)$ to $\mathbb{R}^d$, $\mathcal{C}_d$ denotes the σ-field on C_d generated by the coordinate maps, *i.e.*,

$$\mathcal{C}_d = \sigma\left(\left\{\{\omega(\cdot) \in C_d : \omega(t_1) \in A_1, \cdots, \omega(t_n) \in A_n\} : 0 \leq t_1 < \cdots < t_n < +\infty, A_1, \cdots, A_n \in \mathcal{R}^d\right\}\right),$$

where $\mathcal{R}^d := \mathcal{B}\left(\mathbb{R}^d\right)$ is the Borel σ-field on $\mathbb{R}^d$, and $\mathbb{P}_{\mathbf{x}}$ is the *canonical probability measure* on $(C_d, \mathcal{C}_d)$ so that the continuous-time stochastic process $\{B(t) : t \in \mathbb{R}_+\}$ consists of coordinate maps on $(C_d, \mathcal{C}_d)$ forms a d-dimensional Brownian motion such that $\mathbb{P}_{\mathbf{x}}(\{\omega(\cdot) \in C_d : \omega(0) = \mathbf{x}\}) = 1$. In other words, $\mathbf{x} \in \mathbb{R}^d$ indicates the starting point of $\{B(t) : t \in \mathbb{R}_+\}$ and this d-dimensional Brownian motion is often called the *canonical d-dimensional Brownian motion* [3]. The probability measure $\mathbb{P}_{\mathbf{x}}$ is often called the *Wiener measure* with initial state $\mathbf{x} \in \mathbb{R}^d$, and the canonical probability space $(C_d, \mathcal{C}_d, \mathbb{P}_{\mathbf{x}})$ is referred to as the *Wiener probability space* [2].

Problem 1 (*Exercise 7.2.2.* in [1])**.**

We begin the proof with the following useful result:

Lemma 1. *For every $t \in (0,+\infty)$, we have*

$$\{\omega(\cdot) \in C_1 : T_0(\omega) > t\} = \{\omega(\cdot) \in C_1 := B_s(\omega) = \omega(s) \neq 0,\ \forall s \in (0,t]\}. \tag{1}$$

Proof of Lemma 1.

If $T_0(\omega) = \inf\{s \in (0,+\infty) : B_s(\omega) = \omega(s) = 0\} > t$, it's clear that $B_s(\omega) = \omega(s) \neq 0$ for all $s \in (0,t]$, otherwise we have $T_0(\omega) \leq t$. So the "$\subseteq$" direction is obvious. Now, it remains to prove the "$\supseteq$" direction. If $B_s(\omega) = \omega(s) \neq 0$ for all $s \in (0,t]$, then we know that $\{s \in (0,+\infty) : B_s(\omega) = \omega(s) = 0\} \subseteq (t,+\infty)$ and this yields $T_0(\omega) \geq t$. So it suffices to show that $T_0(\omega) \neq t$. Assume on the contrary that $T_0(\omega) = t$. Then for any given $\epsilon > 0$, there exists an $s(\epsilon) \in [t, t+\epsilon)$ such that $B_{s(\epsilon)}(\omega) = \omega(s(\epsilon)) = 0$. Since $B_t(\omega) = \omega(t) \neq 0$, $s(\epsilon) \in (t, t+\epsilon)$ for every $\epsilon > 0$. Let $s_n := s\left(\frac{1}{n}\right) \in \left(t, t+\frac{1}{n}\right)$ for each $n \in \mathbb{N}$. Then one can see that $\lim_{n\to\infty} s_n = t$, and the continuity of the Brownian path at $\omega(\cdot)$ yields

$$B_t(\omega) = \omega(t) = \omega\left(\lim_{n\to\infty} s_n\right) = \lim_{n\to\infty} \omega(s_n) = \lim_{n\to\infty} \omega\left(s\left(\frac{1}{n}\right)\right) = 0.$$

This fact contradicts the assumption that $B_t(\omega) = \omega(t) \neq 0$, thereby we obtain $T_0(\omega) > t$ and this establishes the "$\supseteq$" direction.

$\square$

We may observe that for any $t \in (0,1)$,

$$\begin{aligned}
\{\omega(\cdot) \in C_1 : L(\omega) > t\} &= \{\omega(\cdot) \in C_1 : B_s(\omega) = \omega(s) = 0 \text{ for some } s \in (t,1]\} \\
&= \theta_t^{-1}\left(\{\omega(\cdot) \in C_1 : B_s(\omega) = \omega(s) = 0 \text{ for some } s \in (0,1-t]\}\right) \\
&= C_1 \setminus \theta_t^{-1}\left(\{\omega(\cdot) \in C_1 : B_s(\omega) = \omega(s) \neq 0,\ \forall s \in (0,1-t]\}\right) \\
&\overset{\text{(a)}}{=} C_1 \setminus \theta_t^{-1}\left(\{\omega(\cdot) \in C_1 : T_0(\omega) > 1-t\}\right) \\
&= \theta_t^{-1}\left(\{\omega(\cdot) \in C_1 : T_0(\omega) \leq 1-t\}\right),
\end{aligned}$$

where the step (a) holds by Lemma 1, and it leads to

$$\{\omega(\cdot) \in C_1 : L(\omega) \leq t\} = \theta_t^{-1}\left(\{\omega(\cdot) \in C_1 : T_0(\omega) > 1-t\}\right)$$

for every $t \in (0,1)$. Thus we arrive at

$$\mathbb{1}_{\{L \leq t\}} = \mathbb{1}_{\theta_t^{-1}(\{T_0 > 1-t\})} = \mathbb{1}_{\{T_0 > 1-t\}} \circ \theta_t, \tag{2}$$

for every $t \in (0,1)$. Hence, we see that for any $t \in (0,1)$,

$$\begin{aligned}
\mathbb{P}_0\{L \leq t\} &= \mathbb{E}_0\left[\mathbb{1}_{\{L\leq t\}}\right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_0\left[\mathbb{1}_{\{T_0 > 1-t\}} \circ \theta_t\right] \\
&= \mathbb{E}_0\left[\mathbb{E}_0\left[\mathbb{1}_{\{T_0>1-t\}} \circ \theta_t \middle| \mathcal{F}_t^+\right]\right] \\
&\overset{\text{(c)}}{=} \mathbb{E}_0\left[\mathbb{E}_{B(t)}\left[\mathbb{1}_{\{T_0>1-t\}}\right]\right] \\
&= \int_{\mathbb{R}} \mathrm{d}y\ p_t(0,y) \cdot \mathbb{E}_y\left[\mathbb{1}_{\{T_0>1-t\}}\right] \\
&= \int_{\mathbb{R}} p_t(0,y) \cdot \mathbb{P}_y\{T_0 > 1-t\}\,\mathrm{d}y,
\end{aligned}$$

where the step (b) follows from the equation (2), and the step (c) is due to the Markov property (*Theorem 7.2.1* in [1]), and this completes the proof.

Problem 2 (*Exercise 7.2.4.* in [1]).

(i) Let $X(f) := \limsup_{t\downarrow 0^+} \frac{B(t)}{f(t)}$, where $f : ((0,+\infty), \mathcal{B}((0,+\infty))) \to ((0,+\infty), \mathcal{B}((0,+\infty)))$ is a measurable function. Then one can see that

$$\begin{aligned} X(f) &= \limsup_{t\downarrow 0^+} \frac{B(t)}{f(t)} \\ &= \lim_{\epsilon\downarrow 0^+} \left[\sup \left\{ \frac{B(t)}{f(t)} : t \in (0,\epsilon) \right\} \right] \\ &= \lim_{\substack{\epsilon\downarrow 0^+: \\ \epsilon\in\left(0,\frac{1}{n}\right]}} \underbrace{\left[\sup \left\{ \frac{B(t)}{f(t)} : t \in (0,\epsilon) \right\} \right]}_{\in\, \mathcal{F}^0_\epsilon \,\subseteq\, \mathcal{F}^+_\epsilon \,\subseteq\, \mathcal{F}^+_{\frac{1}{n}}} \\ &\in \mathcal{F}^+_{\frac{1}{n}} \end{aligned}$$

for every $n \in \mathbb{N}$. Therefore, $X(f)$ is $\left(\bigcap_{n=1}^\infty \mathcal{F}^+_{\frac{1}{n}}\right)$-measurable and the right-continuity of the continuous-time filtration $\{\mathcal{F}_t : t \in R_+\}$ implies $\bigcap_{n=1}^\infty \mathcal{F}^+_{\frac{1}{n}} = \mathcal{F}^+_0$. Hence, $X(f)$ is $\mathcal{F}^+_0$-measurable.

On the other hand, we know that $\mathbb{P}_0\{\tau_+ = 0\} = 1$, where $\tau_+ := \inf\{t \in (0,+\infty) : B(t) > 0\}$, by *Theorem 7.2.4* in [1]. Thus if $\omega(\cdot) \in \{\tau_+ = 0\}$, we have

$$X(f)(\omega) = \lim_{\epsilon\downarrow 0^+} \left[\sup \left\{ \frac{B_t(\omega)}{f(t)} : t \in (0,\epsilon) \right\} \right] \geq 0.$$

So we obtain $\mathbb{P}_0\{X(f) \geq 0\} = 1$. Now, we employ the Blumenthal's zero-one law (*Theorem 7.2.3* in [1]) to the events $\{X(f) \leq \lambda\} \in \mathcal{F}^+_0$: $\mathbb{P}_0\{X(f) \leq \lambda\} \in \{0,1\}$ for every $\lambda \in [0,+\infty]$. Let

$$c := \inf\{\lambda \in [0,+\infty] : \mathbb{P}_0\{X(f) \leq \lambda\} = 1\} \in [0,+\infty].$$

If $c = +\infty$, we have $\mathbb{P}_0\{X(f) = +\infty\} = 1$ and we are done. On the other hand, if $c < +\infty$, it's clear that $\mathbb{P}_0\left\{X(f) \leq c + \frac{1}{n}\right\} = 1$ for all $n \in \mathbb{N}$. So we arrive at

$$\mathbb{P}_0\{X(f) \leq c\} = \lim_{n\to\infty} \downarrow \mathbb{P}_0\left\{X(f) \leq c + \frac{1}{n}\right\} = 1. \tag{3}$$

Also we know that $\mathbb{P}_0\left\{X(f) \leq c - \frac{1}{n}\right\} = 0$ for all $n \in \mathbb{N}$, thereby one has

$$\mathbb{P}_0\{X(f) < c\} = \lim_{n\to\infty} \uparrow \mathbb{P}_0\left\{X(f) \leq c - \frac{1}{n}\right\} = 0. \tag{4}$$

Taking two pieces (3) and (4) collectively yields

$$\mathbb{P}_0\{X(f) = c\} = \mathbb{P}_0\{X(f) \leq c\} - \mathbb{P}_0\{X(f) < c\} = 1 - 0 = 1,$$

i.e., $X(f) = \limsup_{t\downarrow 0^+} \frac{B(t)}{f(t)} = c$, $\mathbb{P}_0$-almost surely, as desired.

(ii) We claim that $\limsup_{t\downarrow 0^+} \frac{B(t)}{\sqrt{t}} = +\infty$, $\mathbb{P}_0$-almost surely. In order to prove this claim, we first consider the continuous-time stochastic process $\{Y(t) : t \in \mathbb{R}_+\}$ on $(C_1, \mathcal{C}_1)$ defined by

$$Y(t) := \begin{cases} tB\left(\frac{1}{t}\right) & \text{if } t \in (0,+\infty); \\ 0 & \text{otherwise.} \end{cases}$$

According to *Theorem 7.2.6* in [1], $\{Y(t) : t \in \mathbb{R}_+\}$ is a one-dimensional Brownian motion starting at $0 \in \mathbb{R}$ under $\mathbb{P}_0$. Also, one has

$$\limsup_{t\downarrow 0^+} \frac{B(t)}{\sqrt{t}} = \limsup_{t\downarrow 0^+} \sqrt{t}Y\left(\frac{1}{t}\right) = \limsup_{s\to+\infty} \frac{Y(s)}{\sqrt{s}}. \tag{5}$$

So it suffices to prove that $\limsup_{s\to+\infty} \frac{Y(s)}{\sqrt{s}} = +\infty$, $\mathbb{P}_0$-almost surely. To this end, we first observe for any $M \in (0, +\infty)$ that

$$\begin{aligned}
\mathbb{P}_0\left\{\limsup_{n\to\infty} \frac{Y(n)}{\sqrt{n}} > M\right\} &= \mathbb{P}_0\left\{\limsup_{n\to\infty} \left\{\frac{Y(n)}{\sqrt{n}} > M\right\}\right\} \\
&= \mathbb{P}_0\left\{\bigcap_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} \left\{\frac{Y(k)}{\sqrt{k}} > M\right\}\right]\right\} \\
&= \lim_{n\to\infty} \downarrow \mathbb{P}_0\left\{\bigcup_{k=n}^{\infty} \left\{\frac{Y(k)}{\sqrt{k}} > M\right\}\right\} \\
&\geq \limsup_{n\to\infty} \mathbb{P}_0\left\{\frac{Y(n)}{\sqrt{n}} > M\right\} \\
&\overset{\text{(a)}}{=} \mathbb{P}\left\{Z > M\right\} > 0,
\end{aligned} \tag{6}$$

where $Z \sim \mathsf{N}(0,1)$, and the step (a) holds since $\frac{Y(n)}{\sqrt{n}} \overset{d}{=} \mathsf{N}(0,1)$ under $\mathbb{P}_0$. We know that for any $k \in \mathbb{N}$,

$$\frac{Y(k)}{\sqrt{k}} = \sqrt{k}B\left(\frac{1}{k}\right)$$

is $\mathcal{F}^+_{\frac{1}{k}}$-measurable. So for each $n \in \mathbb{N}$, $\frac{Y(k)}{\sqrt{k}}$ is $\mathcal{F}^+_{\frac{1}{n}}$-measurable for every $k \geq n$. Therefore $\limsup_{k\to\infty} \frac{Y(k)}{\sqrt{k}}$ is $\mathcal{F}^+_{\frac{1}{n}}$-measurable for every $n \in \mathbb{N}$. From the right-continuity of the continuous-time filtration $\left\{\mathcal{F}^+_t : t \in \mathbb{R}_+\right\}$, we see that $\bigcap_{n=1}^{\infty} \mathcal{F}^+_{\frac{1}{n}} = \mathcal{F}^+_0$ and thus $\limsup_{n\to\infty} \frac{Y(n)}{\sqrt{n}}$ is $\mathcal{F}^+_0$-measurable. Consequently, the Blumenthal's zero-one law (*Theorem 7.2.3* in [1]) together with the inequality (6) implies that

$$\mathbb{P}_0\left\{\limsup_{n\to\infty} \frac{Y(n)}{\sqrt{n}} > M\right\} = 1 \tag{7}$$

for every $M \in (0, +\infty)$. Take

$$\mathcal{A} := \bigcap_{M=1}^{\infty} \left\{\limsup_{n\to\infty} \frac{Y(n)}{\sqrt{n}} > M\right\} \in \mathcal{C}_1.$$

Then we can see that $\mathbb{P}_0\{\mathcal{A}\} = 1$ and if $\omega(\cdot) \in \mathcal{A}$, then

$$\limsup_{n\to\infty} \frac{Y_n(\omega)}{\sqrt{n}} > M \tag{8}$$

for every $M \in \mathbb{N}$. So by letting $M \to +\infty$ in (8), we obtain $\limsup_{n\to\infty} \frac{Y_n(\omega)}{\sqrt{n}} = +\infty$, which also implies

$$\limsup_{s\to+\infty} \frac{Y_s(\omega)}{\sqrt{s}} = +\infty$$

for all $\omega(\cdot) \in \mathcal{A}$. Thus $\limsup_{s\to+\infty} \frac{Y(s)}{\sqrt{s}} = +\infty$, $\mathbb{P}_0$-almost surely, and putting this result into (5) completes the proof.

Problem 3 (*Exercise 7.3.2.* in [1]).

Let $S : (C_d, \mathcal{C}_d) \to ([0,+\infty], \mathcal{B}([0,+\infty]))$ and $T : (C_d, \mathcal{C}_d) \to ([0,+\infty], \mathcal{B}([0,+\infty]))$ be stopping times with respect to the continuous-time filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$. At this point, we recall that the continuous-time filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$ given by

$$\mathcal{F}_t := \bigcap_{\mathbf{x}\in\mathbb{R}^d} \sigma\left(\mathcal{F}_t^+ \cup \mathcal{N}_x\right), \ \forall t \in \mathbb{R}_+,$$

is right-continuous, where $\mathcal{N}_x := \{A \subseteq C_d : A \subseteq B \text{ for some } B \in \mathcal{C}_d \text{ such that } \mathbb{P}_x\{B\} = 0\}$ denotes the collection of all $\mathbb{P}_{\mathbf{x}}$-null sets.

(1) For every $t \in \mathbb{R}_+$, we have

$$\{S \wedge T \leq x\} = \underbrace{\{S \leq x\}}_{\in \mathcal{F}_x} \cup \underbrace{\{T \leq x\}}_{\in \mathcal{F}_x} \in \mathcal{F}_x.$$

Thus, $S \wedge T := \min\{S, T\}$ is a stopping time with respect to the continuous-time filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$.

(2) For every $t \in \mathbb{R}_+$, we have

$$\{S \vee T \leq x\} = \underbrace{\{S \leq x\}}_{\in \mathcal{F}_x} \cap \underbrace{\{T \leq x\}}_{\in \mathcal{F}_x} \in \mathcal{F}_x.$$

Thus, $S \vee T := \max\{S, T\}$ is a stopping time with respect to the continuous-time filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$.

(3) Let $\mathbb{Q}_+ := \mathbb{Q} \cap \mathbb{R}_+ = \mathbb{Q} \cap [0, +\infty)$. We can prove the following useful lemma:

Lemma 2. *For every $x \in \mathbb{R}_+$,*

$$\{S + T < x\} = \bigcup_{(q,r)\in\Gamma(x)} (\{S < q\} \cap \{T < r\}), \tag{9}$$

where $\Gamma(x) := \left\{(q, r) \in \mathbb{Q}_+^2 : q + r < x\right\}$.

Proof of Lemma 2.

The "$\supseteq$" direction is trivial. In order to verify the "$\subseteq$" direction, we choose an $\omega(\cdot) \in \{S + T < x\}$. As $S(\omega) < x - T(\omega)$, we may choose $q(\omega) \in \mathbb{Q} \cap (S(\omega), x - T(\omega))$ and this is possible since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then we have $T(\omega) < x - q(\omega)$, so we may take $r(\omega) \in \mathbb{Q} \cap (T(\omega), x - q(\omega))$ which is also possible since $\mathbb{Q}$ is dense in $\mathbb{R}$. Since $q(\omega) > S(\omega) \geq 0$, $r(\omega) > T(\omega) \geq 0$, and $q(\omega) + r(\omega) < x$, it's clear that $(q(\omega), r(\omega)) \in \Gamma(x)$. Moreover, we see that $S(\omega) < q(\omega)$ and $T(\omega) < r(\omega)$ and this implies that

$$\omega(\cdot) \in \bigcup_{(q,r)\in\Gamma(x)} (\{S < q\} \cap \{T < r\}),$$

and this proves the "$\subseteq$" direction.

□

Note that $\Gamma(x)$ is countable for every $x \in \mathbb{R}_+$, and if $(q, r) \in \Gamma_x$, we have $q \leq q + r < x$ and $r \leq q + r < x$. Thus if $(q, r) \in \Gamma_x$, then

$$\{S < q\} \in \mathcal{F}_q \subseteq \mathcal{F}_x \quad \text{and} \quad \{T < r\} \in \mathcal{F}_r \subseteq \mathcal{F}_x.$$

As a consequence, for every $x \in \mathbb{R}_+$

$$\{S+T<x\} \overset{\text{(a)}}{=} \bigcup_{(q,r)\in\Gamma(x)} \underbrace{(\{S<q\}\cap\{T<r\})}_{\in\,\mathcal{F}_x} \in \mathcal{F}_x, \tag{10}$$

where the step (a) follows from Lemma 2. So,

$$\{S+T\le x\} = \bigcap_{k=n}^{\infty} \underbrace{\left\{S+T<x+\frac{1}{k}\right\}}_{\in\,\mathcal{F}_{x+\frac{1}{k}}\,\in\,\mathcal{F}_{x+\frac{1}{n}}} \in \mathcal{F}_{x+\frac{1}{n}}$$

for all $n \in \mathbb{N}$. Hence, $\{S+T\le x\} \in \bigcap_{n=1}^{\infty}\mathcal{F}_{x+\frac{1}{n}} = \mathcal{F}_x$ owing to the right-continuity of the continuous-time filtration $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$ and this shows that $S+T$ is also a stopping time with respect to $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$.

So in particular, when $t\in\mathbb{R}_+$ is a non-random constant, it's clear that t is a stopping time with respect to $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$. Therefore by employing the above statements (1)–(3), we may conclude that $S\wedge t$, $S\vee t$, and $S+t$ are all stopping times with respect to the right-continuous filtration $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$.

Problem 4 (*Exercise 7.3.3.* in [1])**.**

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of stopping times with respect to the right-continuous filtration $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$.

(1) For every $t\in\mathbb{R}_+$, we see that

$$\{\sup\{T_n : n\in\mathbb{N}\}\le t\} = \bigcap_{n=1}^{\infty} \underbrace{\{T_n\le t\}}_{\in\,\mathcal{F}_t} \in \mathcal{F}_t,$$

thereby $\sup\{T_n : n\in\mathbb{N}\}$ is a stopping time with respect to the right-continuous filtration $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$.

(2) For every $t\in(0,+\infty)$, we see that

$$\{\inf\{T_n : n\in\mathbb{N}\}<t\} = \bigcup_{n=1}^{\infty}\{T_n<t\} = \bigcup_{n=1}^{\infty}\left(\bigcup_{k=1}^{\infty}\underbrace{\left\{T_n\le t-\frac{1}{k}\right\}}_{\in\,\mathcal{F}_{0\vee\left(t-\frac{1}{k}\right)}\,\subseteq\,\mathcal{F}_t}\right) \in \mathcal{F}_t.$$

Therefore, we arrive at

$$\{\inf\{T_n : n\in\mathbb{N}\}\le t\} = \bigcap_{k=m}^{\infty}\underbrace{\left\{\inf\{T_n : n\in\mathbb{N}\}<t+\frac{1}{k}\right\}}_{\in\,\mathcal{F}_{t+\frac{1}{k}}\,\subseteq\,\mathcal{F}_{t+\frac{1}{m}}} \in \mathcal{F}_{t+\frac{1}{m}}$$

for every $m\in\mathbb{N}$, so one has $\{\inf\{T_n : n\in\mathbb{N}\}\le t\} \in \bigcap_{m=1}^{\infty}\mathcal{F}_{t+\frac{1}{m}} = \mathcal{F}_t$ for any $t\in\mathbb{R}_+$ by the right-continuity of the continuous-time filtration $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$. Hence, $\inf\{T_n : n\in\mathbb{N}\}$ is also a stopping time with respect to the right-continuous filtration $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$.

(3) By the statement (1), $\sup\{T_k : k\ge n\}$ is a stopping time with respect to the right-continuous filtration $\{\mathcal{F}_t : t\in\mathbb{R}_+\}$ for every $n\in\mathbb{N}$. Therefore,

$$\limsup_{n\to\infty} T_n = \lim_{n\to\infty}\downarrow(\sup\{T_k : k\ge n\})$$

is also a stopping time with respect to $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$ due to *Theorem 7.3.2* in [1].

(4) By the statement (2), $\inf\{T_k : k \geq n\}$ is a stopping time with respect to the right-continuous filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$ for every $n \in \mathbb{N}$. Therefore,

$$\liminf_{n\to\infty} T_n = \lim_{n\to\infty} \uparrow (\inf\{T_k : k \geq n\})$$

is also a stopping time with respect to $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$ owing to *Theorem 7.3.3* in [1].

Problem 5 (*Exercise 7.3.5.* in [1]**).**

Let S and T be stopping times with respect to the right-continuous filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$. We begin with the following useful lemma:

Lemma 3. *Let $\tau : (C_d, \mathcal{C}_d) \to ([0,+\infty], \mathcal{B}([0,+\infty]))$ be a stopping time with respect to the right-continuous filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$. Then,*

$$\mathcal{F}_\tau = \{A \in \mathcal{C}_d : A \cap \{T < t\} \in \mathcal{F}_t,\ \forall t \in (0,+\infty)\}. \tag{11}$$

Proof of Lemma 3.

For the "$\subseteq$" direction, we choose any $A \in \mathcal{F}_\tau$. Then for every $t \in (0,+\infty)$, one has

$$A \cap \{\tau < t\} = A \cap \left(\bigcup_{n=1}^{\infty} \left\{\tau \leq t - \frac{1}{n}\right\}\right) = \bigcup_{n=1}^{\infty} \underbrace{\left(A \cap \left\{\tau \leq t - \frac{1}{n}\right\}\right)}_{\in\ \mathcal{F}_{0\vee\left(t-\frac{1}{n}\right)}\ \subseteq \mathcal{F}_t} \in \mathcal{F}_t.$$

For the "$\supseteq$" direction, we assume that $A \in \mathcal{C}_d$ is an event such that $A \cap \{\tau < t\} \in \mathcal{F}_t$ for all $t \in (0,+\infty)$. We now fix any $t \in \mathbb{R}_+$. Then we have

$$A \cap \{\tau \leq t\} = A \cap \left(\bigcap_{k=n}^{\infty} \left\{\tau < t + \frac{1}{k}\right\}\right) = \bigcap_{k=n}^{\infty} \underbrace{\left(A \cap \left\{\tau < t + \frac{1}{k}\right\}\right)}_{\in\ \mathcal{F}_{t+\frac{1}{k}}\ \subseteq \in \mathcal{F}_{t+\frac{1}{n}}} \in \mathcal{F}_{t+\frac{1}{n}}$$

for every $n \in \mathbb{N}$. So, $A \cap \{\tau \leq t\} \in \bigcap_{n=1}^{\infty} \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t$ due to the right-continuity of the continuous-time filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$. Hence, $A \in \mathcal{F}_\tau$ and this completes the proof of Lemma 3.

□

(1) We first consider the event $\{S < T\}$.

Claim 1. *For every $s \in (0,+\infty)$, we have*

$$\{S < T\} \cap \{S < s\} = \bigcup_{q \in \mathbb{Q}\cap[0,s]} (\{S < q\} \cap \{T > q\}). \tag{12}$$

Proof of Claim 1.

The direction "$\supseteq$" of the equality (12) is trivial. Now suppose that $S(\omega) < T(\omega)$ and $S(\omega) < s$. Then we have $S(\omega) < \min\{s, T(\omega)\}$ and thus we may choose a rational number $q(\omega) \in \mathbb{Q} \cap (S(\omega), \min\{s, T(\omega)\})$ which is possible since $\mathbb{Q}$ is dense in $\mathbb{R}$. It's clear that $q(\omega) \in \mathbb{Q} \cap [0,s]$ and

$$S(\omega) < q(\omega) < \min\{s, T(\omega)\} \leq s \quad \text{and} \quad T(\omega) \geq \min\{s, T(\omega)\} > q(\omega).$$

Hence, we arrive at $\omega(\cdot) \in \bigcup_{q\in\mathbb{Q}\cap[0,s]} (\{S < q\} \cap \{T > q\})$ and this proves the "$\subseteq$" direction.

□

We note that for every $q \in \mathbb{Q} \cap [0, s]$,

$$\begin{aligned} \{S < q\} &= \bigcup_{n=1}^{\infty} \underbrace{\left\{S \le q - \frac{1}{n}\right\}}_{\in \mathcal{F}_{0 \vee \left(q - \frac{1}{n}\right)} \subseteq \mathcal{F}_q} \in \mathcal{F}_q \subseteq \mathcal{F}_s; \\ \{T > q\} &= C_d \setminus \{T \le q\} \in \mathcal{F}_q \subseteq \mathcal{F}_s. \end{aligned} \tag{13}$$

Thus for any $s \in (0, +\infty)$, one has

$$\{S < T\} \cap \{S < s\} \overset{\text{(a)}}{=} \bigcup_{q \in \mathbb{Q} \cap [0,s]} \underbrace{(\{S < q\} \cap \{T > q\})}_{\in \mathcal{F}_s} \overset{\text{(b)}}{\in} \mathcal{F}_s,$$

where the step (a) is due to Claim 1, and the step (b) follows from the observation (13). So by Lemma 3, we see that $\{S < T\} \in \mathcal{F}_S$.

(2) For every $s \in (0, +\infty)$, we define $\mathcal{I}(s) := \{(q, r) \in \mathbb{Q} \times \mathbb{Q} : 0 \le r < q \le s\}$.

Claim 2. *For every $s \in (0, +\infty)$, we have*

$$\{S > T\} \cap \{S < s\} = \bigcup_{(q,r) \in \mathcal{I}(s)} (\{r < S < q\} \cap \{T < r\}) . \tag{14}$$

Proof of Claim 2.

The direction "$\supseteq$" of the equality (14) is obvious. Suppose $T(\omega) < S(\omega) < s$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can take $q(\omega) \in \mathbb{Q} \cap (S(\omega), s)$ and $r(\omega) \in \mathbb{Q} \cap (T(\omega), S(\omega))$. Then we have

$$0 \le T(\omega) < r(\omega) < S(\omega) < q(\omega) < s,$$

and this implies that $(q(\omega), r(\omega)) \in \mathcal{I}(s)$. Thus $\omega(\cdot) \in \bigcup_{(q,r) \in \mathcal{I}(s)} (\{r < S < q\} \cap \{T < r\})$ and this establishes the "$\subseteq$" direction of the equality (14).

□

One can observe that for every $(q, r) \in \mathcal{I}(s)$,

$$\begin{aligned} \{r < S < q\} &= \{S < q\} \setminus \{S \le r\} = \left(\bigcup_{n=1}^{\infty} \underbrace{\left\{S \le q - \frac{1}{n}\right\}}_{\in \mathcal{F}_{0 \vee \left(q - \frac{1}{n}\right)} \subseteq \mathcal{F}_s} \right) \setminus \underbrace{\{S \le r\}}_{\in \mathcal{F}_r \subseteq \mathcal{F}_s} \in \mathcal{F}_s; \\ \{T < r\} &= \bigcup_{n=1}^{\infty} \underbrace{\left\{T \le r - \frac{1}{n}\right\}}_{\in \mathcal{F}_{0 \vee \left(r - \frac{1}{n}\right)} \subseteq \mathcal{F}_r} \in \mathcal{F}_r \subseteq \mathcal{F}_s. \end{aligned} \tag{15}$$

Thus for every $s \in (0, +\infty)$, one has

$$\{S > T\} \cap \{S < s\} \overset{\text{(c)}}{=} \bigcup_{(q,r) \in \mathcal{I}(s)} \underbrace{(\{r < S < q\} \cap \{T < r\})}_{\in \mathcal{F}_s} \overset{\text{(d)}}{\in} \mathcal{F}_s,$$

where the step (c) holds by Claim 2, and the step (d) makes use of the observation (15). Hence by Lemma 2, $\{S > T\} \in \mathcal{F}_S$.

(3) $\{S = T\} = C_d \setminus (\{S < T\} \cup \{S > T\}) \in \mathcal{F}_S$ by the above statements (1) and (2). Furthermore, by interchanging the roles between S and T, we see that $\{S < T\}$, $\{S > T\}$, and $\{S = T\}$ belong to the stopped σ-field $\mathcal{F}_T$ associated to the stopping time T with respect to the right-continuous filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$. This completes the proof of all the given statements.

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