MAS651 Theory of Stochastic Processes
Homework #7

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May 5, 2021

Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$. Also, we use the symbol $\mathbb{S}$ instead of S to denote the underlying state space of stochastic processes. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space and $\{B(t) : t \in \mathbb{R}_+\}$ refers to a *standard one-dimensional Brownian motion* defined on $(\Omega, \mathcal{F}, \mathbb{P})$, *i.e.*, a one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $B(0) = 0$.

Problem 1 (*Exercise 7.1.1.* in [1]).

We know that $B(s)$ and $B(t) - B(s)$ are independent and

$$B(s) \sim \mathsf{N}(0, s) \quad \text{and} \quad B(t) - B(s) \sim \mathsf{N}(0, t-s),$$

where $\mathsf{N}(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0, +\infty)$. So the joint pdf of $(B(s), B(t) - B(s))$ is given by

$$f(x, y) := \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \cdot \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{y^2}{2(t-s)}\right) = \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left\{-\frac{1}{2}\left(\frac{x^2}{s} + \frac{y^2}{t-s}\right)\right\}$$

for $(x,y)\in\mathbb{R}^2$. Hence, we have for every $0<s<t<+\infty$,

$$\begin{aligned}
\mathbb{P}\{B(s)>0,B(t)>0\} &= \mathbb{P}\{B(s)>0,\{B(t)-B(s)\}+B(s)>0\} \\
&= \int_{\{(x,y)\in\mathbb{R}^2:\ x>0,\ x+y>0\}} f(x,y)\mathrm{d}x\mathrm{d}y \\
&\overset{\text{(a)}}{=} \int_{\left\{(u,v)\in\mathbb{R}^2:\ u>0,\ \sqrt{s}u+\sqrt{t-s}v>0\right\}} f(\sqrt{s}u,\sqrt{t-s}v)\cdot\sqrt{s(t-s)}\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{2\pi}\int_{\left\{(u,v)\in\mathbb{R}^2:\ u>0,\ \sqrt{s}u+\sqrt{t-s}v>0\right\}} \exp\left\{-\frac{1}{2}\left(u^2+v^2\right)\right\}\mathrm{d}u\mathrm{d}v \\
&\overset{\text{(b)}}{=} \frac{1}{2\pi}\int_{\left\{(r,\theta)\in[0,+\infty)\times[0,2\pi):\ r>0,\ -\arctan\left(\sqrt{\frac{s}{t-s}}\right)<\theta<\frac{\pi}{2}\right\}} \exp\left(-\frac{1}{2}r^2\right)\cdot r\mathrm{d}r\mathrm{d}\theta \\
&\overset{\text{(c)}}{=} \frac{1}{2\pi}\left(\int_0^{\infty} r\exp\left(-\frac{1}{2}r^2\right)\mathrm{d}r\right)\left(\int_{-\arctan\left(\sqrt{\frac{s}{t-s}}\right)}^{\frac{\pi}{2}}\mathrm{d}\theta\right) \\
&= \frac{1}{4}+\frac{1}{2\pi}\arctan\left(\sqrt{\frac{s}{t-s}}\right) \\
&= \frac{1}{4}+\frac{1}{2\pi}\arcsin\left(\sqrt{\frac{s}{t}}\right),
\end{aligned}$$

where the steps (a)–(c) can be justified as follows:

(a) change of variables with $(x,y)=\left(\sqrt{s}u,\sqrt{t-s}v\right)$;

(b) change of variables with $(u,v)=(r\cos\theta,r\sin\theta)$;

(c) the Fubini-Tonelli's theorem.

Problem 2 (*Exercise 7.1.4.* in [1]).

We begin with some formal definitions. Let $(\mathbb{S},\mathcal{S})$ be a measurable space and $\mathbb{T}$ be any index set.

Definition 1. Let $\mathbb{S}^{\mathbb{T}}:=\{\omega(\cdot):\mathbb{T}\to\mathbb{S}\}$ and $\mathcal{S}^{\mathbb{T}}$ be the σ-field on $\mathbb{S}^{\mathbb{T}}$ generated by the collection of *cylindrical sets*

$$\left\{\left\{\omega(\cdot)\in\mathbb{S}^{\mathbb{T}}:\omega(t_1)\in A_1,\omega(t_2)\in A_2,\cdots,\omega(t_n)\in A_n\right\}:n\in\mathbb{N},t_1,t_2,\cdots,t_n\in\mathbb{T},A_1,A_2,\cdots,A_n\in\mathcal{S}\right\}.$$

We call $\mathcal{S}^{\mathbb{T}}$ the *product σ-field* or the *cylindrical σ-field* on $\mathbb{S}^{\mathbb{T}}$.

Definition 2. For any subsets $I\subseteq J\subseteq\mathbb{T}$, let $\pi_I^J:\mathbb{S}^J\to\mathbb{S}^I$ denote the canonical projection map $f\mapsto f|_I$. Let $\Sigma(\mathbb{T})$ be the collection of all countable subsets of $\mathbb{T}$. Given any $\mathbb{C}\in\Sigma(\mathbb{T})$ and $E\in\mathcal{S}^{\mathbb{C}}$, we define

$$[\mathbb{C},E]:=\left\{\omega(\cdot)\in\mathbb{S}^{\mathbb{T}}:\pi_{\mathbb{C}}^{\mathbb{T}}(\omega)=(\omega(t):t\in\mathbb{C})\in E\right\}\subseteq\mathbb{S}^{\mathbb{T}}.$$

We say that a subset $A\subseteq\mathbb{S}^{\mathbb{T}}$ has a *countable representation* if $A=[\mathbb{C},E]$ for some $\mathbb{C}\in\Sigma(\mathbb{T})$ and $E\in\mathcal{S}^{\mathbb{C}}$. Let $\mathsf{CR}(\mathbb{S}^{\mathbb{T}})$ denote the collection of all subsets of $\mathbb{S}^{\mathbb{T}}$ which has a countable representation.

Lemma 1. $\mathsf{CR}(\mathbb{S}^{\mathbb{T}})$ *forms a σ-field on* $\mathbb{S}^{\mathbb{T}}$.

Proof of Lemma 1.

To begin with, it's clear that $\varnothing$ and $\mathbb{S}^{\mathbb{T}}$ belong to $\mathsf{CR}(\mathbb{S}^{\mathbb{T}})$.

(i) If $A \in \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$, then $A = [\mathbb{C}, E]$ for some $\mathbb{C} \in \Sigma(\mathbb{T})$ and $E \in \mathcal{S}^{\mathbb{C}}$. Then, $\mathbb{S}^{\mathbb{T}} \setminus A = \left[\mathbb{C}, \mathbb{S}^{\mathbb{C}} \setminus E\right] \in \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$.

(ii) Let $\{A_n\}_{n=1}^{\infty}$ be any sequence in $\mathsf{CR}(\mathbb{S}^{\mathbb{T}})$. Then we may write $A_n = [\mathbb{C}_n, E_n]$ for some $\mathbb{C}_n \in \Sigma(\mathbb{T})$ and $E_n \in \mathcal{S}^{\mathbb{C}_n}$ for each $n \in \mathbb{N}$. Set $\mathbb{C} := \bigcup_{n=1}^{\infty} \mathbb{C}_n \in \Sigma(\mathbb{T})$. Given any subsets $I \subseteq J \subseteq \mathbb{T}$, every pre-image of a cylindrical set in $\mathbb{S}^I$ under the canonical projection map $\pi_I^J : \left(\mathbb{S}^J, \mathcal{S}^J\right) \to \left(\mathbb{S}^I, \mathcal{S}^I\right)$ is also a cylindrical set in $\mathbb{S}^J$. Therefore, the canonical projection map $\pi_I^J : \left(\mathbb{S}^J, \mathcal{S}^J\right) \to \left(\mathbb{S}^I, \mathcal{S}^I\right)$ is measurable. So we find that $\left(\pi_{\mathbb{C}_n}^{\mathbb{C}}\right)^{-1}(E_n) \in \mathcal{S}^{\mathbb{C}}$ for all $n \in \mathbb{N}$. Let $D := \bigcup_{n=1}^{\infty} \left(\pi_{\mathbb{C}_n}^{\mathbb{C}}\right)^{-1}(E_n) \in \mathcal{S}^{\mathbb{C}}$. Then for every $\omega(\cdot) \in \mathbb{S}^{\mathbb{T}}$, the following holds:

$$\begin{aligned}
\omega(\cdot) \in \bigcup_{n=1}^{\infty} A_n \quad &\Leftrightarrow \quad \omega(\cdot) \in A_n \text{ for some } n \in \mathbb{N} \\
&\Leftrightarrow \quad \left(\pi_{\mathbb{C}_n}^{\mathbb{T}}\right)(\omega) \in E_n \text{ for some } n \in \mathbb{N} \\
&\overset{\text{(a)}}{\Leftrightarrow} \quad \left(\pi_{\mathbb{C}}^{\mathbb{T}}\right)(\omega) \in \left(\pi_{\mathbb{C}_n}^{\mathbb{C}}\right)^{-1}(E_n) \text{ for some } n \in \mathbb{N} \\
&\Leftrightarrow \quad \left(\pi_{\mathbb{C}}^{\mathbb{T}}\right)(\omega) \in D \\
&\Leftrightarrow \quad \omega(\cdot) \in [\mathbb{C}, D],
\end{aligned}$$

where the step (a) holds since $\pi_{\mathbb{C}_n}^{\mathbb{T}} = \left(\pi_{\mathbb{C}_n}^{\mathbb{C}}\right) \circ \left(\pi_{\mathbb{C}}^{\mathbb{T}}\right)$ for every $n \in \mathbb{N}$. Hence, we have $\bigcup_{n=1}^{\infty} A_n = [\mathbb{C}, D] \in \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$.

To sum up, we conclude that $\mathsf{CR}(\mathbb{S}^{\mathbb{T}})$ is a σ-field on $\mathbb{S}^{\mathbb{T}}$.

$\square$

Now we claim that $\mathcal{S}^{\mathbb{T}} = \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$. Choose any cylindrical set in $\mathbb{S}^{\mathbb{T}}$,

$$A = \left\{\omega(\cdot) \in \mathbb{S}^{\mathbb{T}} : \omega(t_1) \in A_1, \omega(t_2) \in A_2, \cdots, \omega(t_n) \in A_n\right\},$$

where $t_1, t_2, \cdots, t_n \in \mathbb{T}$ are distinct indices, and $A_1, A_2, \cdots, A_n \in \mathcal{S}$. Let $\mathbb{C}$ be any countable subset of $\mathbb{T}$ containing $\{t_1, t_2, \cdots, t_n\}$, and

$$E := \bigcap_{j=1}^{n} \left(\pi_{\{t_j\}}^{\mathbb{C}}\right)^{-1}(A_j) = \left\{f \in \mathbb{S}^{\mathbb{C}} : f(t_1) \in A_1, f(t_2) \in A_2, \cdots, f(t_n) \in A_n\right\} \in \mathcal{S}^{\mathbb{C}}.$$

Then, $A = [\mathbb{C}, E] \in \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$ and since $\mathsf{CR}(\mathbb{S}^{\mathbb{T}})$ is a σ-field on $\mathbb{S}^{\mathbb{T}}$ ($\because$ Lemma 1) containing all cylindrical sets in $\mathbb{S}^{\mathbb{T}}$, we deduce $\mathcal{S}^{\mathbb{T}} \subseteq \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$.

Conversely, let $A \in \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$. Then,

$$A = [\mathbb{C}, E] = \left(\pi_{\mathbb{C}}^{\mathbb{T}}\right)^{-1}(E) \in \mathcal{S}^{\mathbb{T}},$$

since the canonical projection map $\pi_{\mathbb{C}}^{\mathbb{T}} : \left(\mathbb{S}^{\mathbb{T}}, \mathcal{S}^{\mathbb{T}}\right) \to \left(\mathbb{S}^{\mathbb{C}}, \mathcal{S}^{\mathbb{C}}\right)$ is measurable. Thus, we have $\mathsf{CR}(\mathbb{S}^{\mathbb{T}}) \subseteq \mathcal{S}^{\mathbb{T}}$ and this completes the proof of our desired claim $\mathcal{S}^{\mathbb{T}} = \mathsf{CR}(\mathbb{S}^{\mathbb{T}})$. Employing this result to the case for which $(\mathbb{S}, \mathcal{S}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R}) = \mathcal{R}$ denotes the Borel σ-field on $\mathbb{R}$, and $\mathbb{T} = \mathbb{R}_+ = [0, +\infty)$, we see that $\mathcal{R}^{[0,+\infty)} = \mathcal{F}_0$ and obtain the desired result.

Problem 3 (*Exercise 7.1.5.* in [1]).

To begin with, we fix any $k \in \mathbb{N}$ and an exponent $\gamma \in \left(\frac{1}{2} + \frac{1}{k}, +\infty\right)$. For any given $C, T \in (0, +\infty)$, we consider the event

$$\mathcal{A}_n(C,T) := \left\{\omega \in \Omega : \text{there is an } s \in [0,T] \ \textit{s.t.} \ |B_t(\omega) - B_s(\omega)| \leq C\,|t-s|^\gamma \ \text{ whenever } \ |t-s| \leq k \cdot \frac{T}{n}\right\} \in \mathcal{F},$$

and define

$$Y_{n,l} := \max\left\{\left|B\left((l+j)\cdot\frac{T}{n}\right) - B\left((l+j-1)\cdot\frac{T}{n}\right)\right| : j \in [0:k-1]\right\}$$

for each $l \in [n-k+1]$, and

$$\mathcal{B}_n(C,T) := \left\{\omega \in \Omega : \text{there exists an } l \in [n-k+1] \ \textit{s.t.} \ |Y_{n,l}(\omega)| \leq C\left(\frac{T}{n}\right)^\gamma \{k^\gamma + (k-1)^\gamma\}\right\} \in \mathcal{F}.$$

Claim 1. *For any $C, T \in (0, +\infty)$, we have $\mathcal{A}_n(C,T) \subseteq \mathcal{B}_n(C,T)$.*

Proof of Claim 1.

If $\omega \in \mathcal{A}_n(C,T)$, then there exists an $s \in [0,T]$ such that

$$|B_t(\omega) - B_s(\omega)| \leq C\,|t-s|^\gamma \tag{1}$$

for every $t \in \left[s - k\cdot\frac{T}{n}, s + k\cdot\frac{T}{n}\right]$. Here, we may consider the following two cases:

Case #1. $s \in \left[0, (n-k+1)\cdot\frac{T}{n}\right]$: Then $s \in \left[(l-1)\cdot\frac{T}{n}, l\cdot\frac{T}{n}\right]$ for some $l \in [n-k+1]$. Since we have

$$\begin{aligned}\left|(l+j-1)\cdot\frac{T}{n} - s\right| &\leq j\cdot\frac{T}{n} \leq (k-1)\cdot\frac{T}{n};\\ \left|(l+j)\cdot\frac{T}{n} - s\right| &\leq (j+1)\cdot\frac{T}{n} \leq k\cdot\frac{T}{n},\end{aligned} \tag{2}$$

for every $j \in [0:k-1]$, we obtain from the bound (1) that

$$\begin{aligned}\left|B_{(l+j)\cdot\frac{T}{n}}(\omega) - B_{(l+j-1)\cdot\frac{T}{n}}(\omega)\right| &\overset{\text{(a)}}{\leq} \left|B_{(l+j)\cdot\frac{T}{n}}(\omega) - B_s(\omega)\right| + \left|B_s(\omega) - B_{(l+j-1)\cdot\frac{T}{n}}(\omega)\right|\\ &\leq C\left|(l+j)\cdot\frac{T}{n} - s\right|^\gamma + C\left|s - (l+j-1)\cdot\frac{T}{n}\right|^\gamma\\ &\overset{\text{(b)}}{\leq} C\left(\frac{T}{n}\right)^\gamma\{k^\gamma + (k-1)^\gamma\},\end{aligned}$$

where the step (a) is due to the triangle inequality, and the step (b) follows from the bound (2). Thus we arrive at

$$|Y_{n,l}(\omega)| = \max\left\{\left|B_{(l+j)\cdot\frac{T}{n}}(\omega) - B_{(l+j-1)\cdot\frac{T}{n}}(\omega)\right| : j \in [0:k-1]\right\} \leq C\left(\frac{T}{n}\right)^\gamma\{k^\gamma + (k-1)^\gamma\}.$$

Case #2. $s \in \left((n-k+1)\cdot\frac{T}{n}, T\right]$: Since $(n-k+1+j)\cdot\frac{T}{n} \in \left[(n-k+1)\cdot\frac{T}{n}, T\right]$ and $(n-k+j)\cdot\frac{T}{n} \in \left[(n-k)\cdot\frac{T}{n}, T\right]$ for every $j \in [0:k-1]$, we see that

$$\begin{aligned}\left|(n-k+j)\cdot\frac{T}{n} - s\right| &\leq k\cdot\frac{T}{n};\\ \left|(n-k+1+j)\cdot\frac{T}{n} - s\right| &\leq (k-1)\cdot\frac{T}{n}\end{aligned} \tag{3}$$

for every $j \in [0 : k-1]$. So we may deduce

$$\begin{aligned}\left|B_{(n-k+1+j)\cdot\frac{T}{n}}(\omega) - B_{(n-k+j)\cdot\frac{T}{n}}(\omega)\right| &\overset{\text{(c)}}{\leq} \left|B_{(n-k+1+j)\cdot\frac{T}{n}}(\omega) - B_s(\omega)\right| + \left|B_s(\omega) - B_{(n-k+j)\cdot\frac{T}{n}}(\omega)\right| \\ &\leq C\left|(n-k+1+j)\cdot\frac{T}{n} - s\right|^\gamma + C\left|s - (n-k+j)\cdot\frac{T}{n}\right|^\gamma \\ &\overset{\text{(d)}}{\leq} C\left(\frac{T}{n}\right)^\gamma \left\{k^\gamma + (k-1)^\gamma\right\},\end{aligned}$$

where the step (c) makes use of the triangle inequality, and the step (d) follows from the fact (3). Therefore, one has

$$|Y_{n,n-k+1}(\omega)| = \max\left\{\left|B_{(n-k+1+j)\cdot\frac{T}{n}}(\omega) - B_{(n-k+j)\cdot\frac{T}{n}}(\omega)\right| : j \in [0 : k-1]\right\} \leq C\left(\frac{T}{n}\right)^\gamma \left\{k^\gamma + (k-1)^\gamma\right\}.$$

To sum up, we conclude that $\omega \in \mathcal{B}_n(C, T)$ as desired.

□

Thanks to Claim 1, we arrive at

$$\begin{aligned}&\mathbb{P}\left\{\mathcal{A}_n(C,T)\right\} \\ &\leq \mathbb{P}\left\{\mathcal{B}_n(C,T)\right\} \\ &= \mathbb{P}\left\{\bigcup_{l=1}^{n-k+1}\left\{\max\left\{\left|B\left((l+j)\cdot\frac{T}{n}\right) - B\left((l+j-1)\cdot\frac{T}{n}\right)\right| : j \in [0:k-1]\right\} \leq C\left(\frac{T}{n}\right)^\gamma \left\{k^\gamma + (k-1)^\gamma\right\}\right\}\right\} \\ &\overset{\text{(e)}}{\leq} \sum_{l=1}^{n-k+1}\mathbb{P}\left\{\bigcap_{j=0}^{k-1}\left\{\left|B\left((l+j)\cdot\frac{T}{n}\right) - B\left((l+j-1)\cdot\frac{T}{n}\right)\right| \leq C\left(\frac{T}{n}\right)^\gamma \left\{k^\gamma + (k-1)^\gamma\right\}\right\}\right\} \\ &\overset{\text{(f)}}{=} \sum_{l=1}^{n-k+1}\left[\prod_{j=0}^{k-1}\mathbb{P}\left\{\left|B\left((l+j)\cdot\frac{T}{n}\right) - B\left((l+j-1)\cdot\frac{T}{n}\right)\right| \leq C\left(\frac{T}{n}\right)^\gamma \left\{k^\gamma + (k-1)^\gamma\right\}\right\}\right] \\ &\overset{\text{(g)}}{=} \sum_{l=1}^{n-k+1}\left[\prod_{j=0}^{k-1}\mathbb{P}\left\{\left|B\left(\frac{T}{n}\right)\right| \leq C\left(\frac{T}{n}\right)^\gamma \left\{k^\gamma + (k-1)^\gamma\right\}\right\}\right] \\ &\overset{\text{(h)}}{\leq} n\left[\mathbb{P}\left\{|Z| \leq C\left(\frac{T}{n}\right)^{\gamma-\frac{1}{2}}\left\{k^\gamma + (k-1)^\gamma\right\}\right\}\right]^k \\ &\leq n\left[\int_{-C\left(\frac{T}{n}\right)^{\gamma-\frac{1}{2}}\left\{k^\gamma+(k-1)^\gamma\right\}}^{C\left(\frac{T}{n}\right)^{\gamma-\frac{1}{2}}\left\{k^\gamma+(k-1)^\gamma\right\}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)\mathrm{d}x\right]^k \\ &\overset{\text{(i)}}{\leq} n\left[\frac{2C\left\{k^\gamma + (k-1)^\gamma\right\}\left(\frac{T}{n}\right)^{\gamma-\frac{1}{2}}}{\sqrt{2\pi}}\right]^k \\ &= \left[\frac{2C\left\{k^\gamma + (k-1)^\gamma\right\}T^{\gamma-\frac{1}{2}}}{\sqrt{2\pi}}\right]^k \cdot n^{-k\left(\gamma-\frac{1}{2}-\frac{1}{k}\right)},\end{aligned} \tag{4}$$

where $Z \sim \mathsf{N}(0,1)$, and the above steps (e)–(i) can be validated as follows:

(e) the union bound;

(f) $\{B(t) : t \in \mathbb{R}_+\}$ has independent increments;

(g) $B\left((l+j)\cdot\frac{T}{n}\right) - B\left((l+j-1)\cdot\frac{T}{n}\right) \stackrel{d}{=} B\left(\frac{T}{n}\right) \sim \mathsf{N}\left(0, \frac{T}{n}\right)$ for every $l \in [n-k+1]$ and $j \in [0:k-1]$;

(h) $B\left(\frac{T}{n}\right) \stackrel{d}{=} \sqrt{\frac{T}{n}}Z$;

(i) $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \leq \frac{1}{\sqrt{2\pi}}$ for all $x \in \mathbb{R}$.

Since the exponent γ satisfies $\gamma > \frac{1}{2} + \frac{1}{k}$, by letting $n \to \infty$ in the bound (4) we obtain for every $C, T > 0$,

$$0 = \lim_{n\to\infty} \mathbb{P}\left\{\mathcal{A}_n(C,T)\right\} \stackrel{\text{(j)}}{=} \mathbb{P}\left\{\bigcup_{n=1}^{\infty} \mathcal{A}_n(C,T)\right\}, \tag{5}$$

where the step (j) follows from the fact that $\{\mathcal{A}_n(C,T)\}_{n=1}^{\infty}$ is a non-decreasing sequence of events with respect to the set inclusion.

Finally, we claim that for $\mathbb{P}$-a.s. $\omega \in \Omega$, the sample path $t \in [0,T] \mapsto B_t(\omega) \in \mathbb{R}$ of the one-dimensional Brownian motion $\{B(t) : t \in \mathbb{R}_+\}$ at ω, *i.e.*, the Brownian path at ω is not γ-Hölder continuous at any point in $[0,T]$, where $\gamma \in \left(\frac{1}{2}+\frac{1}{k}, +\infty\right)$. We consider the following event for each $T > 0$, given by

$$\mathcal{E}(T) := \{\omega \in \Omega : \text{the Brownian path at } \omega \text{ is } \gamma\text{-Hölder continuous at some point } s \in [0,T]\} \in \mathcal{F}.$$

Then, one can see for every $t > 0$ that

$$\mathcal{E}(T) \subseteq \bigcup_{n=1}^{\infty}\left[\bigcup_{M=1}^{\infty} \mathcal{A}_n(M,T)\right].$$

Due to the countable sub-additivity of $\mathbb{P}\{\cdot\}$, one has

$$\begin{aligned}
\mathbb{P}\left\{\mathcal{E}(T)\right\} &\leq \mathbb{P}\left\{\bigcup_{n=1}^{\infty}\left[\bigcup_{M=1}^{\infty} \mathcal{A}_n(M,T)\right]\right\} \\
&= \mathbb{P}\left\{\bigcup_{M=1}^{\infty}\left[\bigcup_{n=1}^{\infty} \mathcal{A}_n(M,T)\right]\right\} \\
&\leq \sum_{M=1}^{\infty} \mathbb{P}\left\{\bigcup_{n=1}^{\infty} \mathcal{A}_n(M,T)\right\} \\
&\stackrel{\text{(k)}}{=} 0,
\end{aligned}$$

where the step (k) is due to the result (5), thereby $\mathbb{P}\left\{\Omega \setminus \mathcal{E}(T)\right\} = 1$. Also, it's clear that if $\omega \in \Omega \setminus \mathcal{E}(T)$, the Brownian path $t \in [0,T] \mapsto B_t(\omega) \in \mathbb{R}$ is not γ-Hölder continuous at every point in $[0,T]$ and this completes the proof of our claim. Since this claim holds for any $T > 0$, it suffices to put $T = 1$ in order to achieve the desired result in this problem.

Problem 4 (*Exercise 7.1.6.* in [1]).

Since $\{B(t) : t \in \mathbb{R}_+\}$ has independent increments and it satisfies the following properties:

$$B(0) = 0 \quad \text{and} \quad B(s+t) - B(s) \sim \mathsf{N}(0,t) \text{ if } s \geq 0 \text{ and } t > 0,$$

we see that for any fixed $t \in (0, +\infty)$,

$$\left\{\Delta_{m,n} := B\left(\frac{t}{2^n}m\right) - B\left(\frac{t}{2^n}(m-1)\right) : m \in [2^n]\right\} \stackrel{\text{i.i.d.}}{\sim} \mathsf{N}\left(0, \frac{t}{2^n}\right). \tag{6}$$

Thus we have

$$\mathbb{E}\left[\sum_{m=1}^{2^n} \Delta_{m,n}^2\right] = \sum_{m=1}^{2^n} \mathbb{E}\left[\Delta_{m,n}^2\right] = \sum_{m=1}^{2^n} \mathsf{Var}\left[\Delta_{m,n}\right] = t. \tag{7}$$

Therefore, we arrive at

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2 - \mathbb{E}\left[\sum_{m=1}^{2^n} \Delta_{m,n}^2\right]\right)^2\right] \\
&= \mathsf{Var}\left[\sum_{m=1}^{2^n} \Delta_{m,n}^2\right] \\
&\overset{\text{(a)}}{=} \sum_{m=1}^{2^n} \mathsf{Var}\left[\Delta_{m,n}^2\right] \\
&\overset{\text{(b)}}{=} 2^n \cdot \mathsf{Var}\left[\Delta_{1,n}^2\right] \\
&= 2^n \left\{\mathbb{E}\left[\Delta_{1,n}^4\right] - \left(\mathbb{E}\left[\Delta_{1,n}^2\right]\right)^2\right\} \\
&\overset{\text{(c)}}{=} 2^n \left\{\mathbb{E}\left[\Delta_{1,n}^4\right] - \left(\frac{t}{2^n}\right)^2\right\} \\
&\overset{\text{(d)}}{=} 2^n \left\{\left(\sqrt{\frac{t}{2^n}}\right)^4 \mathbb{E}\left[Z^4\right] - \left(\frac{t}{2^n}\right)^2\right\} \\
&= \frac{t^2}{2^n}\left(\mathbb{E}\left[Z^4\right] - 1\right) \\
&\overset{\text{(e)}}{=} \frac{t^2}{2^{n-1}},
\end{aligned} \tag{8}$$

where $Z \sim \mathsf{N}(0,1)$, and the above steps (a)–(e) can be verified as follows:

(a) the independence of $\{\Delta_{m,n} : m \in [2^n]\}$;

(b) $\{\Delta_{m,n} : m \in [2^n]\}$ is identically distributed (see (7) for further details);

(c) $\Delta_{1,n} \sim \mathsf{N}\left(0, \frac{t}{2^n}\right)$;

(d) the same reason as the step (c);

(e) we can see via a simple integration by parts that $\mathbb{E}\left[Z^4\right] = 3$.

Hereafter, we let $X_n := \sum_{m=1}^{2^n} \Delta_{m,n}^2$ for each $n \in \mathbb{N}$. From the computations (7) and (8), we know that

$$\mathbb{E}\left[X_n\right] = t \quad \text{and} \quad \mathsf{Var}\left[X_n\right] = \frac{t^2}{2^{n-1}}.$$

Let $\mathcal{D}_n(\epsilon) := \{\omega \in \Omega : |X_n(\omega) - t| > \epsilon\} \in \mathcal{F}$ for any $\epsilon > 0$. Then we obtain for every $n \in \mathbb{N}$,

$$\mathbb{P}\left\{\mathcal{D}_n(\epsilon)\right\} \overset{\text{(f)}}{\leq} \epsilon^{-2} \cdot \mathbb{E}\left[\left|X_n(\omega) - t\right|^2\right] = \epsilon^{-2} \cdot \mathsf{Var}\left[X_n\right] = \left(\frac{t}{\epsilon}\right)^2 \cdot \frac{1}{2^{n-1}},$$

where the step (f) follows from the Chebyshev's inequality. Thus, it leads to

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\mathcal{D}_n(\epsilon)\right\} \leq \left(\frac{t}{\epsilon}\right)^2 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2\left(\frac{t}{\epsilon}\right)^2 < +\infty,$$

thereby the first Borel-Cantelli lemma implies

$$\mathbb{P}\left\{\limsup_{n\to\infty}\mathcal{D}_n(\epsilon)\right\}=0,\ \forall\epsilon>0. \tag{9}$$

Now, we define

$$\mathcal{E}:=\bigcap_{k=1}^{\infty}\left\{\Omega\setminus\left(\limsup_{n\to\infty}\mathcal{D}_n\left(\frac{1}{k}\right)\right)\right\}\in\mathcal{F}.$$

Since

$$\mathbb{P}\left\{\Omega\setminus\mathcal{E}\right\}=\mathbb{P}\left\{\bigcup_{k=1}^{\infty}\limsup_{n\to\infty}\mathcal{D}_n\left(\frac{1}{k}\right)\right\}\le\sum_{k=1}^{\infty}\mathbb{P}\left\{\limsup_{n\to\infty}\mathcal{D}_n\left(\frac{1}{k}\right)\right\}=0,$$

we find that $\mathbb{P}\left\{\mathcal{E}\right\}=1$. Moreover, if $\omega\in\mathcal{E}$, we see that for every $k\in\mathbb{N}$,

$$|X_n(\omega)-t|\le\frac{1}{k}$$

for all but finitely many $n\in\mathbb{N}$. Thus we arrive at

$$\limsup_{n\to\infty}|X_n(\omega)-t|\le\frac{1}{k},\ \forall k\in\mathbb{N}, \tag{10}$$

thereby letting $k\to\infty$ in the right-hand side of (10) yields $\lim_{n\to\infty}|X_n(\omega)-t|=0$. Hence,

$$\lim_{n\to\infty}\sum_{m=1}^{2^n}\Delta_{m,n}^2(\omega)=\lim_{n\to\infty}X_n(\omega)=t$$

for all $\omega\in\mathcal{E}$, and this shows that

$$\sum_{m=1}^{2^n}\Delta_{m,n}^2\overset{n\to\infty}{\longrightarrow}t$$

$\mathbb{P}$-almost surely, as desired.

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.