MAS651 Theory of Stochastic Processes
Homework #6

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$. Also, we use the symbol $\mathbb{S}$ instead of S to denote the underlying state space of stochastic processes.

Problem 1 (*Exercise 6.1.1.* in [1])**.**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ be a measure-preserving transformation, and $\mathcal{I}_\varphi$ denote the collection of all φ-invariant events, *i.e.*,

$$\mathcal{I}_\varphi := \left\{A \in \mathcal{F} : \mathbb{P}\left\{A \triangle \varphi^{-1}(A)\right\} = 0\right\}.$$

Then, it's clear that (i) $\varnothing$ and Ω belong to $\mathcal{I}_\varphi$. Choose any $A \in \mathcal{I}_\varphi$. Since $\varphi^{-1}(\Omega \setminus A) = \Omega \setminus \varphi^{-1}(A)$,

$$(\Omega \setminus A) \triangle \varphi^{-1}(\Omega \setminus A) = (\Omega \setminus A) \triangle \left(\Omega \setminus \varphi^{-1}(A)\right) = A \triangle \varphi^{-1}(A),$$

so $\mathbb{P}\left\{(\Omega \setminus A) \triangle \varphi^{-1}(\Omega \setminus A)\right\} = \mathbb{P}\left\{A \triangle \varphi^{-1}(A)\right\} = 0$. Thus, (ii) $A \in \mathcal{I}_\varphi$ implies $\Omega \setminus A \in \mathcal{I}_\varphi$. Now, choose any sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{I}_\varphi$. From

$$\begin{aligned}
\left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \varphi^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \left(\bigcup_{n=1}^{\infty} \varphi^{-1}(A_n)\right) \\
&= \left[\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus \left(\bigcup_{n=1}^{\infty} \varphi^{-1}(A_n)\right)\right] \cup \left[\left(\bigcup_{n=1}^{\infty} \varphi^{-1}(A_n)\right) \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)\right] \\
&= \left[\bigcup_{n=1}^{\infty} \left\{A_n \setminus \left(\bigcup_{n=1}^{\infty} \varphi^{-1}(A_n)\right)\right\}\right] \cup \left[\bigcup_{n=1}^{\infty} \left\{\varphi^{-1}(A_n) \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)\right\}\right] \\
&\subseteq \left[\bigcup_{n=1}^{\infty} \left\{A_n \setminus \varphi^{-1}(A_n)\right\}\right] \cup \left[\bigcup_{n=1}^{\infty} \left\{\varphi^{-1}(A_n) \setminus A_n\right\}\right] \\
&= \bigcup_{n=1}^{\infty} \underbrace{\left[\left\{A_n \setminus \varphi^{-1}(A_n)\right\} \cup \left\{\varphi^{-1}(A_n) \setminus A_n\right\}\right]}_{= A_n \triangle \varphi^{-1}(A_n)} \\
&= \bigcup_{n=1}^{\infty} \left\{A_n \triangle \varphi^{-1}(A_n)\right\},
\end{aligned}$$

one can see that

$$\mathbb{P}\left\{\left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \varphi^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right)\right\} \overset{\text{(a)}}{\leq} \sum_{n=1}^{\infty} \mathbb{P}\left\{A_n \triangle \varphi^{-1}(A_n)\right\} = 0,$$

where the step (a) follows from the countable sub-additivity of $\mathbb{P}\{\cdot\}$. Therefore, (iii) if $\{A_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{I}_\varphi$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{I}_\varphi$. So, the above properties (i), (ii), and (iii) shows that $\mathcal{I}_\varphi$ is a σ-field on Ω.

Now, let's prove that a random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{I}_\varphi$-measurable if and only if $X = X \circ \varphi$ $\mathbb{P}$-almost surely. (#) We first prove the "only if" part: assume that X is $\mathcal{I}_\varphi$-measurable. For any $q \in \mathbb{Q}$, let

$$\begin{aligned}\mathcal{E}_q &:= X^{-1}\left((-\infty, q]\right) \triangle \left(X \circ \varphi\right)^{-1}\left((-\infty, q]\right) \\ &= \left\{\omega \in \Omega : \text{either } X(\omega) \leq q, \left(X \circ \varphi\right)(\omega) > q \text{ or } X(\omega) > q, \left(X \circ \varphi\right)(\omega) \leq q\right\}.\end{aligned}$$

Since X is $\mathcal{I}_\varphi$-measurable, $X^{-1}\left((-\infty, q]\right) \in \mathcal{I}_\varphi$ for all $q \in \mathbb{Q}$ and thus

$$\mathbb{P}\left\{\mathcal{E}_q\right\} = \mathbb{P}\left\{X^{-1}\left((-\infty, q]\right) \triangle \varphi^{-1}\left(X^{-1}\left((-\infty, q]\right)\right)\right\} = 0, \ \forall q \in \mathbb{Q}.$$

At this point, we claim that

$$\left\{\omega \in \Omega : X(\omega) \neq \left(X \circ \varphi\right)(\omega)\right\} = \bigcup_{q \in \mathbb{Q}} \mathcal{E}_q. \tag{1}$$

We may observe that for any $B \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}&X^{-1}(B) \triangle \left(X \circ \varphi\right)^{-1}(B) \\ =\ &\left\{\omega \in \Omega : \text{either } X(\omega) \in B, \left(X \circ \varphi\right)(\omega) \in \mathbb{R} \setminus B \text{ or } X(\omega) \in \mathbb{R} \setminus B, \left(X \circ \varphi\right)(\omega) \in B\right\} \\ \subseteq\ &\left\{\omega \in \Omega : X(\omega) \neq \left(X \circ \varphi\right)(\omega)\right\}.\end{aligned} \tag{2}$$

The observation (2) implies $\mathcal{E}_q \subseteq \left\{\omega \in \Omega : X(\omega) \neq \left(X \circ \varphi\right)(\omega)\right\}$ for every $q \in \mathbb{Q}$, thereby one has

$$\bigcup_{q \in \mathbb{Q}} \mathcal{E}_q \subseteq \left\{\omega \in \Omega : X(\omega) \neq \left(X \circ \varphi\right)(\omega)\right\}. \tag{3}$$

On the other hand, if $\omega \in \{X \neq X \circ \varphi\}$, we may choose a rational number $q \in \mathbb{Q}$ lying between $X(\omega)$ and $\left(X \circ \varphi\right)(\omega)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. For this case, we have either $X(\omega) < q < \left(X \circ \varphi\right)(\omega)$ or $\left(X \circ \varphi\right)(\omega) < q < X(\omega)$ and this implies

$$\begin{aligned}\omega &\in \left\{\omega \in \Omega : \text{either } X(\omega) \leq q, \left(X \circ \varphi\right)(\omega) > q \text{ or } X(\omega) > q, \left(X \circ \varphi\right)(\omega) \leq q\right\} \\ &= X^{-1}\left((-\infty, q]\right) \triangle \left(X \circ \varphi\right)^{-1}\left((-\infty, q]\right) \\ &= \mathcal{E}_q.\end{aligned}$$

Therefore, we arrive at

$$\left\{\omega \in \Omega : X(\omega) \neq \left(X \circ \varphi\right)(\omega)\right\} \subseteq \bigcup_{q \in \mathbb{Q}} \mathcal{E}_q, \tag{4}$$

and combining (3) together with (4) yields our desired claim (1). Hence, we can see that

$$\mathbb{P}\left\{\omega \in \Omega : X(\omega) \neq \left(X \circ \varphi\right)(\omega)\right\} = \mathbb{P}\left\{\bigcup_{q \in \mathbb{Q}} \mathcal{E}_q\right\} \overset{\text{(b)}}{\leq} \sum_{q \in \mathbb{Q}} \mathbb{P}\left\{\mathcal{E}_q\right\} \overset{\text{(c)}}{=} 0, \tag{5}$$

where the step (b) is due to the countable sub-additivity of $\mathbb{P}\{\cdot\}$, and the step (c) holds since $\mathbb{P}\left\{\mathcal{E}_q\right\} = 0$ for all $q \in \mathbb{Q}$, and the result (5) shows that if X is $\mathcal{I}_\varphi$-measurable, then $X = X \circ \varphi$ $\mathbb{P}$-almost surely.

Finally, we prove the "if" part of the statement (#). Suppose that X is *φ-invariant*, *i.e.*, $X = X \circ \varphi$ $\mathbb{P}$-almost surely. The set relation (2) implies

$$\mathbb{P}\left\{X^{-1}(B) \triangle (X \circ \varphi)^{-1}(B)\right\} = \mathbb{P}\left\{X^{-1}(B) \triangle \varphi^{-1}\left(X^{-1}(B)\right)\right\} = 0$$

for all $B \in \mathcal{B}(\mathbb{R})$, and thus $X^{-1}(B) \in \mathcal{I}_\varphi$ for every $B \in \mathcal{B}(\mathbb{R})$. Hence, X is $\mathcal{I}_\varphi$-measurable and this completes the proof of the statement (#).

Problem 2 (*Exercise 6.1.2.* in [1]).

(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ be a measure-preserving transformation, and $\mathcal{I}_\varphi$ denote the *φ-invariant σ-field*. Note that we have seen that $\mathcal{I}_\varphi$ forms a σ-field on Ω. Now, let $A \in \mathcal{F}$ and $B := \bigcup_{n=0}^{\infty} \varphi^{-n}(A) \in \mathcal{F}$. Then,

$$\varphi^{-1}(B) = \varphi^{-1}\left\{\bigcup_{n=0}^{\infty} \varphi^{-n}(A)\right\} = \bigcup_{n=0}^{\infty} \varphi^{-1}\left\{\varphi^{-n}(A)\right\} = \bigcup_{n=0}^{\infty} \varphi^{-(n+1)}(A) \subseteq \bigcup_{n=0}^{\infty} \varphi^{-n}(A) = B.$$

(ii) Let $B \in \mathcal{F}$ be any event such that $\varphi^{-1}(B) \subseteq B$ and $C := \bigcap_{n=0}^{\infty} \varphi^{-n}(B) \in \mathcal{F}$. Then,

$$\begin{aligned}\varphi^{-1}(C) &= \{\omega \in \Omega : \varphi(\omega) \in C\} \\ &= \{\omega \in \Omega : \varphi^{n}(\varphi(\omega)) \in B \text{ for all } n \in \mathbb{Z}_+\} \\ &= \bigcap_{n=1}^{\infty} \varphi^{-n}(B).\end{aligned} \tag{6}$$

Thus, one has

$$C = \bigcap_{n=0}^{\infty} \varphi^{-n}(B) = B \cap \left[\bigcap_{n=1}^{\infty} \varphi^{-n}(B)\right] \overset{\text{(a)}}{=} B \cap \varphi^{-1}(C) \overset{\text{(b)}}{=} \varphi^{-1}(C),$$

where the step (a) is simply the equality (6), and the step (b) holds since

$$\varphi^{-1}(C) = \varphi^{-1}\left\{\bigcap_{n=0}^{\infty} \varphi^{-n}(B)\right\} \subseteq \varphi^{-1}(B) \subseteq B,$$

and this establishes the statement (ii).

(iii) Let us begin with the following useful lemma:

Lemma 1. *If $A \in \mathcal{I}_\varphi$, then $\mathbb{P}\left\{A \triangle \varphi^{-n}(A)\right\} = 0$ for all $n \in \mathbb{Z}_+$.*

Proof of Lemma 1.

The case $n = 0$ and $n = 1$ is trivial from the definition of the φ-invariance. So, we assume $n \geq 2$.

$$\begin{aligned}\mathbb{P}\left\{A \setminus \varphi^{-n}(A)\right\} &= \mathbb{P}\left\{\left(A \setminus \varphi^{-(n-1)}(A)\right) \setminus \varphi^{-n}(A)\right\} + \mathbb{P}\left\{\left(A \cap \varphi^{-(n-1)}(A)\right) \setminus \varphi^{-n}(A)\right\} \\ &\leq \mathbb{P}\left\{A \setminus \varphi^{-(n-1)}(A)\right\} + \mathbb{P}\left\{\varphi^{-(n-1)}(A) \setminus \varphi^{-n}(A)\right\} \\ &= \mathbb{P}\left\{A \setminus \varphi^{-(n-1)}(A)\right\} + \mathbb{P}\left\{\varphi^{-(n-1)}\left(A \setminus \varphi^{-1}(A)\right)\right\} \\ &\overset{\text{(c)}}{=} \mathbb{P}\left\{A \setminus \varphi^{-(n-1)}(A)\right\} + \mathbb{P}\left\{A \setminus \varphi^{-1}(A)\right\},\end{aligned} \tag{7}$$

where the step (c) follows from the fact that $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ is measure-preserving. Analogously,

$$\begin{aligned}\mathbb{P}\left\{\varphi^{-n}(A) \setminus A\right\} &= \mathbb{P}\left\{\left(\varphi^{-n}(A) \setminus \varphi^{-(n-1)}(A)\right) \setminus A\right\} + \mathbb{P}\left\{\left(\varphi^{-n}(A) \cap \varphi^{-(n-1)}(A)\right) \setminus A\right\} \\ &\leq \mathbb{P}\left\{\varphi^{-n}(A) \setminus \varphi^{-(n-1)}(A)\right\} + \mathbb{P}\left\{\varphi^{-(n-1)}(A) \setminus A\right\} \\ &= \mathbb{P}\left\{\varphi^{-(n-1)}\left(\varphi^{-1}(A) \setminus A\right)\right\} + \mathbb{P}\left\{\varphi^{-(n-1)}(A) \setminus A\right\} \\ &\overset{\text{(d)}}{=} \mathbb{P}\left\{\varphi^{-1}(A) \setminus A\right\} + \mathbb{P}\left\{\varphi^{-(n-1)}(A) \setminus A\right\},\end{aligned} \tag{8}$$

where the step (d) holds by the fact that $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ is measure-preserving. By summing two inequalities (7) and (8), we arrive at

$$\mathbb{P}\left\{A \triangle \varphi^{-n}(A)\right\} \leq \mathbb{P}\left\{A \triangle \varphi^{-(n-1)}(A)\right\} + \underbrace{\mathbb{P}\left\{A \triangle \varphi^{-1}(A)\right\}}_{=\,0} = \mathbb{P}\left\{A \triangle \varphi^{-(n-1)}(A)\right\} \tag{9}$$

for every $n \geq 2$. So, we may inductively deduce

$$\mathbb{P}\left\{A \triangle \varphi^{-n}(A)\right\} \leq \mathbb{P}\left\{A \triangle \varphi^{-1}(A)\right\} = 0$$

for all $n \geq 2$, and this establishes our desired result.

□

Now, it's time to prove the following statement: $A \in \mathcal{I}_\varphi$ if and only if there is an event $C \in \mathcal{F}$ such that $\varphi^{-1}(C) = C$ and $\mathbb{P}\{A \triangle C\} = 0$. We first prove the "only if" part of the statement. Given any $A \in \mathcal{I}_\varphi$, let

$$C := \limsup_{n \to \infty} \varphi^{-n}(A) = \bigcap_{n=0}^{\infty} \left[\bigcup_{k=n}^{\infty} \varphi^{-k}(A)\right] = \bigcap_{n=0}^{\infty} \varphi^{-n}\left(\bigcup_{k=0}^{\infty} \varphi^{-k}(A)\right) \in \mathcal{F}.$$

From the above statements (i) and (ii), we see that $C = \varphi^{-1}(C)$, *i.e.*, C is φ-invariant in the strict sense. Now, it remains to prove $\mathbb{P}\{A \triangle C\} = 0$. We reach

$$\begin{aligned}\mathbb{P}\{A \setminus C\} &= \mathbb{P}\left\{A \cap \left[\bigcup_{n=0}^{\infty}\left\{\bigcap_{k=n}^{\infty} \varphi^{-k}(\Omega \setminus A)\right\}\right]\right\} \\ &= \mathbb{P}\left\{\bigcup_{n=0}^{\infty}\left[A \cap \left\{\bigcap_{k=n}^{\infty} \varphi^{-k}(\Omega \setminus A)\right\}\right]\right\} \\ &\overset{\text{(e)}}{\leq} \sum_{n=0}^{\infty} \mathbb{P}\left\{A \cap \left\{\bigcap_{k=n}^{\infty} \varphi^{-k}(\Omega \setminus A)\right\}\right\} \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}\left\{A \cap \varphi^{-n}(\Omega \setminus A)\right\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\left\{A \setminus \varphi^{-n}(A)\right\},\end{aligned} \tag{10}$$

where the step (e) comes from the countable sub-additivity of $\mathbb{P}\{\cdot\}$, and

$$\begin{aligned}\mathbb{P}\{C \setminus A\} &= \mathbb{P}\left\{\left[\bigcap_{n=0}^{\infty}\left\{\bigcup_{k=n}^{\infty} \varphi^{-k}(A)\right\}\right] \setminus A\right\} \\ &\leq \mathbb{P}\left\{\left\{\bigcup_{n=0}^{\infty} \varphi^{-n}(A)\right\} \setminus A\right\} \\ &\overset{\text{(f)}}{\leq} \sum_{n=0}^{\infty} \mathbb{P}\left\{\varphi^{-n}(A) \setminus A\right\},\end{aligned} \tag{11}$$

where the step (f) is owing to the countable sub-additivity of $\mathbb{P}\{\cdot\}$. Summing two bounds (10) and (11) yields

$$\mathbb{P}\{A\triangle C\} \le \sum_{n=0}^{\infty} \left(\mathbb{P}\left\{A \setminus \varphi^{-n}(A)\right\} + \mathbb{P}\left\{\varphi^{-n}(A) \setminus A\right\}\right) = \sum_{n=0}^{\infty} \mathbb{P}\left\{A\triangle\varphi^{-n}(A)\right\} \overset{\text{(g)}}{=} 0,$$

where the step (g) makes use of Lemma 1, and this establishes the "only if" part of our target statement.

Finally, it remains to prove the "if" part of the statement. One has

$$\begin{aligned}
0 &= \mathbb{P}\{A\triangle C\} \\
&= \mathbb{P}\{A \setminus C\} + \mathbb{P}\{C \setminus A\} \\
&\overset{\text{(h)}}{=} \mathbb{P}\left\{\varphi^{-1}(A \setminus C)\right\} + \mathbb{P}\left\{\varphi^{-1}(C \setminus A)\right\} \\
&= \mathbb{P}\left\{\varphi^{-1}(A) \setminus \varphi^{-1}(C)\right\} + \mathbb{P}\left\{\varphi^{-1}(C) \setminus \varphi^{-1}(A)\right\} \\
&\overset{\text{(i)}}{=} \mathbb{P}\left\{\varphi^{-1}(A) \setminus C\right\} + \mathbb{P}\left\{C \setminus \varphi^{-1}(A)\right\} \\
&= \mathbb{P}\left\{\varphi^{-1}(A)\triangle C\right\},
\end{aligned}$$

where the step (h) follows from the fact that φ is measure-preserving, and the step (i) is due to the assumption $C = \varphi^{-1}(C)$. From the following two inequalities

$$\begin{aligned}
&\mathbb{P}\left\{A \setminus \varphi^{-1}(A)\right\} = \mathbb{P}\left\{(A \cap C) \setminus \varphi^{-1}(A)\right\} + \mathbb{P}\left\{(A \setminus C) \setminus \varphi^{-1}(A)\right\} \le \mathbb{P}\left\{C \setminus \varphi^{-1}(A)\right\} + \mathbb{P}\{A \setminus C\}; \\
&\mathbb{P}\left\{\varphi^{-1}(A) \setminus A\right\} = \mathbb{P}\left\{\left(\varphi^{-1}(A) \cap C\right) \setminus A\right\} + \mathbb{P}\left\{\left(\varphi^{-1}(A) \setminus C\right) \setminus A\right\} \le \mathbb{P}\{C \setminus A\} + \mathbb{P}\left\{\varphi^{-1}(A) \setminus C\right\},
\end{aligned}$$

we have

$$\mathbb{P}\left\{A\triangle\varphi^{-1}(A)\right\} \le \mathbb{P}\{A\triangle C\} + \mathbb{P}\left\{\varphi^{-1}(A)\triangle C\right\} = 0.$$

Hence, $A \in \mathcal{I}_\varphi$, *i.e.*, A is $\mathbb{P}$-*almost* φ-*invariant* and this completes the proof of the "if" part of the statement.

Problem 3 (*Exercise 6.1.4.* in [1]).

Let $\{X_n : (\Omega, \mathcal{F}) \to (\mathbb{S}, \mathcal{S})\}_{n=0}^{\infty}$ be a stationary sequence defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $(\mathbb{S}, \mathcal{S})$ is a nice measurable space, *i.e.*, a standard Borel space. In order to establish the desired result, we first provide the following modification of *Kolmogorov's extension theorem*:

Lemma 2. *Let $(\mathbb{S}, \mathcal{S})$ be a nice measurable space, T be a countable index set, and $\Sigma(T)$ denote the collection of all finite subsets of T. For any subsets $I \subseteq J \subseteq T$, let $\pi_I^J : \mathbb{S}^J \to \mathbb{S}^I$, $\omega \mapsto \omega|_I$, be the canonical projection map. Suppose $\{\mu_I : I \in \Sigma(T)\}$ is a collection of probability measures, where μ_I is a probability measure on $\left(\mathbb{S}^I, \mathcal{S}^I\right)$ for $I \in \Sigma(T)$, satisfying the following consistency property: for finite subsets $F \subseteq G \subseteq T$, we have $\mu_F = \mu_G \circ \left(\pi_F^G\right)^{-1}$, i.e.,*

$$\mu_F(A) = \mu_G\left\{\left(\pi_F^G\right)^{-1}(A)\right\}, \ \forall A \in \mathcal{S}^F. \tag{12}$$

Then, there is a unique probability measure μ on the product space $\left(\mathbb{S}^T, \mathcal{S}^T\right)$ such that $\mu_F = \mu \circ \left(\pi_F^T\right)^{-1}$ for all $F \in \Sigma(T)$. Here, we make use of the following convention: given any measurable function $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ between two measurable spaces, we define $f^{-1} : \mathcal{B} \to \mathcal{A}$ to be $B \in \mathcal{B} \mapsto f^{-1}(B) \in \mathcal{A}$.

Proof of Lemma 2.

Let $T := \{t_n : n \in \mathbb{N}\}$ be the enumeration of T, and $T_n := \{t_1, t_2, \cdots, t_n\}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let ν_n be the probability measure on $(\mathbb{S}^n, \mathcal{S}^n)$ defined by

$$\nu_n(A_1 \times A_2 \times \cdots \times A_n) := \mu_{T_n}\left(\left\{\omega \in \mathbb{S}^{T_n} : \omega_{t_1} \in A_1, \omega_{t_2} \in A_2, \cdots, \omega_{t_n} \in A_n\right\}\right)$$

for $A_1, A_2, \cdots, A_n \in \mathcal{S}$, and then extend it to $\mathcal{S}^n$ via the π-λ theorem (*Theorem 2.1.6* in [1]). Then one can see that for every $n \geq 2$,

$$\begin{aligned}
\nu_{n-1}(A_1 \times A_2 \times \cdots \times A_{n-1}) &= \mu_{T_{n-1}}\left(\left\{\omega \in \mathbb{S}^{T_{n-1}} : \omega_{t_1} \in A_1, \omega_{t_2} \in A_2, \cdots, \omega_{t_{n-1}} \in A_{n-1}\right\}\right) \\
&\overset{\text{(a)}}{=} \left(\mu_{T_n} \circ \left(\pi^{T_n}_{T_{n-1}}\right)^{-1}\right)\left(\left\{\omega \in \mathbb{S}^{T_{n-1}} : \omega_{t_1} \in A_1, \omega_{t_2} \in A_2, \cdots, \omega_{t_{n-1}} \in A_{n-1}\right\}\right) \\
&= \mu_{T_n}\left(\left\{\omega \in \mathbb{S}^{T_n} : \omega_{t_1} \in A_1, \omega_{t_2} \in A_2, \cdots, \omega_{t_{n-1}} \in A_{n-1}, \omega_{t_n} \in \mathbb{S}\right\}\right) \\
&= \nu_n((A_1 \times A_2 \times \cdots \times A_{n-1}) \times \mathbb{S})
\end{aligned}$$

for all $A_1, A_2, \cdots, A_{n-1} \in \mathcal{S}$, and this implies that $\{\nu_n\}_{n=1}^{\infty}$ is a consistent sequence of probability measures. According to the *original Kolmogorov's extension theorem*, there exists a unique probability measure ν on $\left(\mathbb{S}^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}}\right)$ such that

$$\nu\left(\left\{\omega \in \mathbb{S}^{\mathbb{N}} : \omega_1 \in A_1, \omega_2 \in A_2, \cdots, \omega_n \in A_n\right\}\right) = \nu_n(A_1 \times A_2 \times \cdots \times A_n)$$

for every $n \in \mathbb{N}$ and $A_1, A_2, \cdots, A_n \in \mathcal{S}$. Now, let $\Phi : \left(\mathbb{S}^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}}\right) \to \left(\mathbb{S}^T, \mathcal{S}^T\right)$ be a measurable map defined by

$$[\Phi(\omega)]_{t_n} := \omega_n, \ \forall n \in \mathbb{N},$$

for every $\omega \in \mathbb{S}^{\mathbb{N}}$, and $\mu(A) := \nu\left\{\Phi^{-1}(A)\right\}$ for $A \in \mathcal{S}^T$. Now it only remains to verify $\mu_F = \mu \circ \left(\pi^T_F\right)^{-1}$ for every $F \in \Sigma(T)$. Since F is a finite subset of T, we have $F \subseteq T_k$ for some $k \in \mathbb{N}$ and thus $\mu_F = \mu_{T_k} \circ \left(\pi^{T_k}_F\right)^{-1}$ from the condition (12). It leads to

$$\begin{aligned}
\mu_F\left(\prod_{i \in F} A_i\right) &= \mu_{T_k}\left\{\left(\pi^{T_k}_F\right)^{-1}\left(\prod_{i \in I} A_i\right)\right\} \\
&= \mu_{T_k}\left(\prod_{i \in T_k} \tilde{A}_i\right) \\
&= \mu_{T_k}\left(\left\{\omega \in \mathbb{S}^{T_k} : \omega_{t_1} \in \tilde{A}_{t_1}, \omega_{t_2} \in \tilde{A}_{t_2}, \cdots, \omega_{t_k} \in \tilde{A}_{t_k}\right\}\right) \\
&= \nu_k\left(\prod_{j=1}^{k} \tilde{A}_{t_j}\right) \\
&\overset{\text{(b)}}{=} \nu\left(\left\{\omega \in \mathbb{S}^{\mathbb{N}} : \omega_1 \in \tilde{A}_{t_1}, \omega_2 \in \tilde{A}_{t_2}, \cdots, \omega_k \in \tilde{A}_{t_k}\right\}\right) \\
&\overset{\text{(c)}}{=} \nu\left(\Phi^{-1}\left(\left\{\omega \in \mathbb{S}^T : \omega_{t_1} \in \tilde{A}_{t_1}, \omega_{t_2} \in \tilde{A}_{t_2}, \cdots, \omega_{t_k} \in \tilde{A}_{t_k}\right\}\right)\right) \\
&= \mu\left(\left\{\omega \in \mathbb{S}^T : \omega_{t_1} \in \tilde{A}_{t_1}, \omega_{t_2} \in \tilde{A}_{t_2}, \cdots, \omega_{t_k} \in \tilde{A}_{t_k}\right\}\right) \\
&= \mu\left(\left\{\omega \in \mathbb{S}^T : \omega_i \in A_i, \ \forall i \in F\right\}\right) \\
&= \left(\mu \circ \left(\pi^T_F\right)^{-1}\right)\left(\prod_{i \in I} A_i\right)
\end{aligned}$$

for every $A_i \in \mathcal{S}$, $i \in F$, where the step (b) follows from the construction of the canonical probability measure ν on $\left(\mathbb{S}^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}}\right)$, and the step (c) is due to the definition of the measurable map $\Phi : \left(\mathbb{S}^{\mathbb{N}} \mathcal{S}^{\mathbb{N}}\right) \to \left(\mathbb{S}^T, \mathcal{S}^T\right)$. We note that we employed the convention

$$\tilde{A}_i := \begin{cases} A_i & \text{if } i \in F; \\ \mathbb{S} & \text{otherwise,} \end{cases}$$

for every $i \in T_k$, and the π-λ theorem (*Theorem 2.1.6* in [1]) deduces $\mu_F = \mu \circ \left(\pi_F^T\right)^{-1}$ on the whole σ-field $\mathcal{S}^F$. This completes the proof of Lemma 2.

□

Now, we consider the case for which $T = \mathbb{Z}$ in Lemma 2. Let $\Sigma(\mathbb{Z})$ be the collection of all finite subsets of $\mathbb{Z}$. Given any $I \in \Sigma(\mathbb{Z})$, where $I := \{i_1, i_2, \cdots, i_n\}$ with $i_1 < i_2 < \cdots < i_n$, we define a probability measure μ_I on $\left(\mathbb{S}^I, \mathcal{S}^I\right)$ by

$$\mu_I\left(\prod_{i\in I} A_i\right) := \mathbb{P}\left\{X_{i_1+i_1^-} \in A_{i_1}, X_{i_2+i_1^-} \in A_{i_2}, \cdots, X_{i_n+i_1^-} \in A_{i_n}\right\} \tag{13}$$

for $A_{i_1}, A_{i_2}, \cdots, A_{i_n} \in \mathcal{S}$, and extend it to $\mathcal{S}^I$ via the π-λ theorem, where $x^- := \max\{0, -x\}$ for $x \in \mathbb{R}$. Then for every $k \in [n]$, putting $A_{i_k} = \mathbb{S}$ to (13) yields

$$\begin{aligned}
\mu_{I\setminus\{i_k\}}\left(\prod_{i\in I\setminus\{i_k\}} A_i\right) &= \mathbb{P}\left\{X_{i_l+i_1^-} \in A_{i_l},\ \forall l \in [n]\setminus\{k\}\right\} \\
&= \mu_I\left(A_{i_1} \times \cdots \times A_{i_{k-1}} \times \mathbb{S} \times A_{i_{k+1}} \times \cdots \times A_{i_n}\right) \\
&= \mu_I\left\{\left(\pi^I_{I\setminus\{i_k\}}\right)^{-1}\left(\prod_{i\in I\setminus\{i_k\}} A_i\right)\right\}
\end{aligned}$$

for every $A_i \in \mathcal{S}$, $i \in I \setminus \{i_k\}$, and thus the uniqueness part of Kolmogorov's extension theorem yields

$$\mu_{I\setminus\{i_k\}} = \mu_I \circ \left(\pi^I_{I\setminus\{i_k\}}\right)^{-1} \tag{14}$$

for every $k \in [n]$. So for any finite subsets $J \subseteq I \subseteq \mathbb{Z}$, one can see that $\mu_J = \mu_I \circ \left(\pi_J^I\right)^{-1}$ by employing (14) repeatedly, and this implies that the collection $\{\mu_I : I \in \Sigma(\mathbb{Z})\}$ of probability measures, where μ_I is a probability measure on $\left(\mathbb{S}^I, \mathcal{S}^I\right)$ for $I \in \Sigma(\mathbb{Z})$, satisfies the consistency property (12). According to Lemma 2, there is a unique probability measure μ on $\left(\mathbb{S}^\mathbb{Z}, \mathcal{S}^\mathbb{Z}\right)$ such that $\mu_F = \mu \circ \left(\pi_F^\mathbb{Z}\right)^{-1}$ for all $F \in \Sigma(\mathbb{Z})$. Now, let $Y_n : \mathbb{S}^\mathbb{Z} \to \mathbb{S}$ be defined by $Y_n(\omega) := \omega_n$ for every $n \in \mathbb{Z}$. Choose any $k \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$. Then for any $A_0, A_1, \cdots, A_m \in \mathcal{S}$, one has

$$\begin{aligned}
&\mu\left(\{Y_k \in A_0, Y_{k+1} \in A_1, \cdots, Y_{k+m} \in A_m\}\right) \\
&= \mu\left(\left\{\omega \in \mathbb{S}^\mathbb{Z} : \omega_k \in A_0, \omega_{k+1} \in A_1, \cdots, \omega_{k+m} \in A_m\right\}\right) \\
&= \mu\left(\left(\pi^\mathbb{Z}_{[k:k+m]}\right)^{-1}\left(\left\{\omega \in \mathbb{S}^{[k:k+m]} : \omega_k \in A_0, \omega_{k+1} \in A_1, \cdots, \omega_{k+m} \in A_m\right\}\right)\right) \\
&= \mu_{[k:k+m]}\left(\left\{\omega \in \mathbb{S}^{[k:k+m]} : \omega_k \in A_0, \omega_{k+1} \in A_1, \cdots, \omega_{k+m} \in A_m\right\}\right) \\
&\overset{\text{(d)}}{=} \mathbb{P}\left\{X_{k+k^-} \in A_0, X_{(k+1)+k^-} \in A_1, \cdots, X_{(k+m)+k^-} \in A_m\right\} \\
&\overset{\text{(e)}}{=} \mathbb{P}\left\{X_0 \in A_0, X_1 \in A_1, \cdots, X_m \in A_m\right\},
\end{aligned} \tag{15}$$

where the step (d) comes from the definition of $\mu_{[k:k+m]}$ as (13), and the step (e) holds since the stochastic process $\{X_n\}_{n=0}^\infty$ is a stationary sequence defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Since the equation (15) holds for any $k \in \mathbb{Z}$, we may conclude that the sequence of random vectors $\{(Y_k, Y_{k+1}, \cdots, Y_{k+m}) : k \in \mathbb{Z}\}$ are identically distributed under μ constructed on $\left(\mathbb{S}^\mathbb{Z}, \mathcal{S}^\mathbb{Z}\right)$, for every $m \in \mathbb{Z}_+$. Hence, $\{Y_n : n \in \mathbb{Z}\}$ is a two-sided stationary sequence defined on the canonical probability space $\left(\mathbb{S}^\mathbb{Z}, \mathcal{S}^\mathbb{Z}, \mu\right)$ and this finishes the proof.

Problem 4 (*Exercise 6.2.1.* in [1]).

Since the function $\Phi_p(x) := |x|^p$, $x \in \mathbb{R}$, is a convex function for $p > 1$, we have

$$\mathbb{E}[|X|]^p = \Phi_p(\mathbb{E}[|X|]) \overset{(a)}{\leq} \mathbb{E}[\Phi_p(|X|)] = \mathbb{E}[|X|^p] \overset{(b)}{<} +\infty,$$

where the step (a) follows from Jensen's inequality, and the step (b) holds since $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$, and this implies $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. So, the Birkhoff's ergodic theorem (*Theorem 6.2.1* in [1]) implies

$$\frac{1}{n}\sum_{k=0}^{n-1} X \circ \varphi^k \overset{n\to\infty}{\longrightarrow} \mathbb{E}[X|\mathcal{I}_\varphi] \tag{16}$$

$\mathbb{P}$-almost surely, where $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ is a given measure-preserving transformation and $\mathcal{I}_\varphi$ denotes the φ-invariant σ-field on Ω. Since $\mathbb{E}[X|\mathcal{I}_\varphi]$ is $\mathcal{I}_\varphi$-measurable, Problem 1 implies $\mathbb{E}[X|\mathcal{I}_\varphi] = \mathbb{E}[X|\mathcal{I}_\varphi] \circ \varphi^k$ $\mathbb{P}$-almost surely, for every $k \in \mathbb{Z}_+$. Thus, one has

$$\frac{1}{n}\sum_{k=0}^{n-1} X \circ \varphi^k - \mathbb{E}[X|\mathcal{I}_\varphi] \overset{\mathbb{P}\text{-a.s.}}{=} \frac{1}{n}\sum_{k=0}^{n-1} (X - \mathbb{E}[X|\mathcal{I}_\varphi]) \circ \varphi^k, \tag{17}$$

and (16) implies $\frac{1}{n}\sum_{k=0}^{n-1}(X - \mathbb{E}[X|\mathcal{I}_\varphi]) \circ \varphi^k \overset{n\to\infty}{\longrightarrow} 0$ $\mathbb{P}$-almost surely. Let $X' := X - \mathbb{E}[X|\mathcal{I}_\varphi]$, and note that $X' \in L^p(\Omega, \mathcal{F}, \mathbb{P})$, since

$$\mathbb{E}[|\mathbb{E}[X|\mathcal{I}_\varphi]|^p] \overset{(c)}{\leq} \mathbb{E}[\mathbb{E}[|X|^p|\mathcal{I}_\varphi]] = \mathbb{E}[|X|^p] < +\infty,$$

where the step (c) makes use of Jensen's inequality. Thanks to the equality (17), it suffices to prove

$$\frac{1}{n}\sum_{k=0}^{n-1} X' \circ \varphi^k \overset{L^p}{\longrightarrow} 0 \tag{18}$$

as $n \to \infty$.

Given any $M > 0$, let $Y_M := X' \cdot \mathbb{1}_{\{|X'| \leq M\}}$ and $Z_M := X' - Y_M = X' \cdot \mathbb{1}_{\{|X'| > M\}}$. Then,

$$\frac{1}{n}\sum_{k=0}^{n-1} X' \circ \varphi^k \overset{\mathbb{P}\text{-a.s.}}{=} \underbrace{\left\{\left(\frac{1}{n}\sum_{k=0}^{n-1} Y_M \circ \varphi^k\right) - \mathbb{E}[Y_M|\mathcal{I}_\varphi]\right\}}_{=:\ (\mathrm{T1})} + \underbrace{\left\{\left(\frac{1}{n}\sum_{k=0}^{n-1} Z_M \circ \varphi^k\right) - \mathbb{E}[Z_M|\mathcal{I}_\varphi]\right\}}_{=:\ (\mathrm{T2})}, \tag{19}$$

since $\mathbb{E}[Y_M|\mathcal{I}_\varphi] + \mathbb{E}[Z_M|\mathcal{I}_\varphi] \overset{\mathbb{P}\text{-a.s.}}{=} \mathbb{E}[X'|\mathcal{I}_\varphi] \overset{\mathbb{P}\text{-a.s.}}{=} 0$. From the convexity of the function $\Phi_p(\cdot)$, one has

$$\frac{1}{n}\sum_{k=1}^{n} |x_k|^p = \frac{1}{n}\sum_{k=1}^{n} \Phi_p(x_k) \geq \Phi_p\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) = \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right|^p \tag{20}$$

for every $n \in \mathbb{N}$ and $x_1, x_2, \cdots, x_n \in \mathbb{R}$. Employing the inequality (20) for the case $n = 2$ to the decomposition (19) and then taking expectations gives

$$\mathbb{E}\left[\left|\frac{1}{n}\sum_{k=0}^{n-1} X' \circ \varphi^k\right|^p\right] = \mathbb{E}[|(\mathrm{T1}) + (\mathrm{T2})|^p] \leq 2^{p-1}\left(\mathbb{E}[|(\mathrm{T1})|^p] + \mathbb{E}[|(\mathrm{T2})|^p]\right). \tag{21}$$

So, in order to prove our claim (18), we need to show that $\mathbb{E}[|(\mathrm{T1})|^p] \overset{n\to\infty}{\longrightarrow} 0$ and $\mathbb{E}[|(\mathrm{T2})|^p] \overset{n\to\infty}{\longrightarrow} 0$.

Due to the Birkhoff's ergodic theorem (*Theorem 6.2.1* in [1]), one has

$$(\mathrm{T1}) = \left(\frac{1}{n}\sum_{k=0}^{n-1} Y_M \circ \varphi^k\right) - \mathbb{E}[Y_M|\mathcal{I}_\varphi] \overset{n\to\infty}{\longrightarrow} 0 \tag{22}$$

$\mathbb{P}$-almost surely. Note that $Y_M \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ and thus $Y_M \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. On the other hand,

$$\begin{aligned}
|(\mathrm{T1})|^p &= \left| \left(\frac{1}{n} \sum_{k=0}^{n-1} Y_M \circ \varphi^k \right) - \mathbb{E}\left[Y_M | \mathcal{I}_\varphi\right] \right|^p \\
&= \left| \frac{1}{n} \sum_{k=0}^{n-1} \left(Y_M \circ \varphi^k - \mathbb{E}\left[Y_M | \mathcal{I}_\varphi\right] \right) \right|^p \\
&\overset{\text{(d)}}{\leq} \frac{1}{n} \sum_{k=0}^{n-1} \left| Y_M \circ \varphi^k - \mathbb{E}\left[Y_M | \mathcal{I}_\varphi\right] \right|^p \\
&\overset{\text{(e)}}{\leq} \frac{1}{n} \sum_{k=0}^{n-1} \left(|Y_M| \circ \varphi^k + \left|\mathbb{E}\left[Y_M | \mathcal{I}_\varphi\right]\right| \right)^p \\
&\overset{\text{(f)}}{\leq} \frac{1}{n} \sum_{k=0}^{n-1} \left(|Y_M| \circ \varphi^k + \mathbb{E}\left[|Y_M| \,| \mathcal{I}_\varphi\right] \right)^p \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} (2M)^p = (2M)^p
\end{aligned}$$

$\mathbb{P}$-almost surely, where the above steps can be justified as follows:

(d) the inequality (20);

(e) the triangle inequality;

(f) Jensen's inequality for conditional expectations.

Therefore, the bounded convergence theorem together with the $\mathbb{P}$-almost sure convergence (22) yields

$$\lim_{n\to\infty} \mathbb{E}\left[|(\mathrm{T1})|^p\right] = \mathbb{E}\left[\lim_{n\to\infty} |(\mathrm{T1})|^p\right] = 0. \tag{23}$$

Now, let's take a closer look at the second term (T2). Applying the inequality (17) for the case $n = 2$ to $(T2)$ and then taking expectations yields

$$\mathbb{E}\left[|(\mathrm{T2})|^p\right] \leq 2^{p-1} \left(\mathbb{E}\left[\left| \frac{1}{n} \sum_{k=0}^{n-1} Z_M \circ \varphi^k \right|^p \right] + \mathbb{E}\left[\left| \mathbb{E}\left[Z_M | \mathcal{I}_\varphi\right] \right|^p \right] \right). \tag{24}$$

One can see that

$$\begin{aligned}
\mathbb{E}\left[\left| \frac{1}{n} \sum_{k=0}^{n-1} Z_M \circ \varphi^k \right|^p \right] &\overset{\text{(g)}}{\leq} \mathbb{E}\left[\frac{1}{n} \sum_{k=0}^{n-1} \left| Z_M \circ \varphi^k \right|^p \right] \\
&= \mathbb{E}\left[\frac{1}{n} \sum_{k=0}^{n-1} |Z_M|^p \circ \varphi^k \right] \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[|Z_M|^p \circ \varphi^k \right] \\
&\overset{\text{(h)}}{=} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[|Z_M|^p\right] \\
&= \mathbb{E}\left[|Z_M|^p\right],
\end{aligned} \tag{25}$$

where the step (g) makes use of the inequality (17), and the step (h) follows from the following lemma:

Lemma 3. *If $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ is a measure-preserving transformation, then we have*

$$\mathbb{E}[X] = \mathbb{E}\left[X \circ \varphi^k\right], \ \forall k \in \mathbb{Z}_+, \tag{26}$$

for any non-negative random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Proof of Lemma 3.

First we prove that the statement (26) holds for any $\mathcal{F}$-measurable indicator function. Let $A \in \mathcal{F}$. Then,

$$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}\{A\} \overset{\text{(i)}}{=} \mathbb{P}\left\{\varphi^{-k}(A)\right\} = \mathbb{E}\left[\mathbb{1}_{\varphi^{-k}(A)}\right] = \mathbb{E}\left[\mathbb{1}_A \circ \varphi^k\right],$$

where the step (i) holds since $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ is measure-preserving, thereby the statement (26) holds for the case $X = \mathbb{1}_A$. So the statement (26) also holds for any $\mathcal{F}$-measurable simple function due to the linearlity of expectations. Now, we take a non-decreasing sequence $\{\Gamma_n\}_{n=0}^{\infty}$ of non-negative, $\mathcal{F}$-measurable simple functions such that $X = \lim_{n\to\infty} \uparrow \Gamma_n$ on Ω, where $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a non-negative random variable. Since $X \circ \varphi^k = \lim_{n\to\infty} \uparrow \Gamma_n \circ \varphi^k$ on Ω, the monotone convergence theorem implies

$$\mathbb{E}[X] = \lim_{n\to\infty} \uparrow \mathbb{E}[\Gamma_n] = \lim_{n\to\infty} \uparrow \mathbb{E}\left[\Gamma_n \circ \varphi^k\right] = \mathbb{E}\left[X \circ \varphi^k\right],$$

as desired.

□

Moreover, one has

$$\mathbb{E}\left[|\mathbb{E}[Z_M|\mathcal{I}_\varphi]|^p\right] = \mathbb{E}\left[\Phi_p\left(\mathbb{E}[Z_M|\mathcal{I}_\varphi]\right)\right] \overset{\text{(j)}}{\le} \mathbb{E}\left[\mathbb{E}[\Phi_p(Z_M)|\mathcal{I}_\varphi]\right] = \mathbb{E}[\Phi_p(Z_M)] = \mathbb{E}[|Z_M|^p], \tag{27}$$

where the step (j) follows from Jensen's inequality for conditional expectations. Putting two pieces (25) and (27) into (24) deduces

$$\mathbb{E}\left[|(\mathrm{T2})|^p\right] \le 2^{p-1} \cdot 2\mathbb{E}[|Z_M|^p] = 2^p \cdot \mathbb{E}\left[|X'|^p \mathbb{1}_{\{|X'|>M\}}\right]. \tag{28}$$

As the final step, we combine two pieces (23) and (28): from the bound (21), we arrive at

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}\left[\left|\frac{1}{n}\sum_{k=0}^{n-1} X' \circ \varphi^k\right|^p\right] &\le 2^{p-1}\left[\left\{\limsup_{n\to\infty} \mathbb{E}[|(\mathrm{T1})|^p]\right\} + \left\{\limsup_{n\to\infty} \mathbb{E}[|(\mathrm{T2})|^p]\right\}\right] \\ &\le 2^{p-1} \cdot 2^p \cdot \mathbb{E}\left[|X'|^p \mathbb{1}_{\{|X'|>M\}}\right] \\ &= 2^{2p-1} \cdot \mathbb{E}\left[|X'|^p \mathbb{1}_{\{|X'|>M\}}\right] \end{aligned} \tag{29}$$

for every $M > 0$. Letting $M \to +\infty$ in the bound (29) yields

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{1}{n}\sum_{k=0}^{n-1} X' \circ \varphi^k\right|^p\right] = 0$$

due to the dominated convergence theorem ($\because X' \in L^p(\Omega, \mathcal{F}, \mathbb{P})$), and this leads to our desired result (18).

Problem 5 (*Exercise 6.2.2.* in [1]).

(i) To begin with, let $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ be a measure-preserving transformation, $\mathcal{I}_\varphi$ denote the φ-invariant σ-field on Ω, and $\{g_n\}_{n=0}^{\infty}$ be a sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $g_n \xrightarrow{\mathbb{P}\text{-a.s.}} g$ as $n \to \infty$, where $g \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{E}[\sup\{|g_n| : n \in \mathbb{Z}_+\}] < +\infty$. For this set-up, we prove the following crucial result:

Claim 1.

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} g_k\circ\varphi^k \overset{\mathbb{P}\text{-a.s.}}{\leq} \mathbb{E}\left[g|\mathcal{I}_\varphi\right]. \tag{30}$$

Proof of Claim 1.

For each $N\in\mathbb{Z}_+$, we define a random variable $h_N := \sup\{|g_n - g| : n\geq N\}$, which is $\mathbb{P}$-almost surely finite since

$$\mathbb{E}\left[|h_N|\right] = \mathbb{E}\left[\sup\{|g_n-g| : n\geq N\}\right] \leq \mathbb{E}\left[(\sup\{|g_n| : n\in\mathbb{Z}_+\})\right] + \mathbb{E}\left[|g|\right] < +\infty.$$

Then for every $n > N$,

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} g_k\circ\varphi^k &= \frac{1}{n}\sum_{k=0}^{N-1} g_k\circ\varphi^k + \frac{1}{n}\sum_{k=N}^{n-1} g_k\circ\varphi^k \\ &\overset{\text{(a)}}{\leq} \frac{1}{n}\sum_{k=0}^{N-1}(g+h_0)\circ\varphi^k + \frac{1}{n}\sum_{k=N}^{n-1}(g+h_N)\circ\varphi^k \\ &= \frac{1}{n}\sum_{k=0}^{N-1}(h_0-h_N)\circ\varphi^k + \frac{1}{n}\sum_{k=0}^{n-1}(g+h_N)\circ\varphi^k\end{aligned} \tag{31}$$

$\mathbb{P}$-almost surely, where the step (a) holds since $|g_i - g|\leq h_0$ for every $i\in[0:N-1]$ and $|g_j-g|\leq h_N$ for every $j\in[N:n-1]$. By letting $n\to\infty$ in the bound (31), we can see from the Birkhoff's ergodic theorem (*Theorem 6.2.1* in [1]) that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} g_k\circ\varphi^k \overset{\mathbb{P}\text{-a.s.}}{\leq} \mathbb{E}\left[g+h_N|\mathcal{I}_\varphi\right] \overset{\mathbb{P}\text{-a.s.}}{=} \mathbb{E}\left[g|\mathcal{I}_\varphi\right] + \mathbb{E}\left[h_N|\mathcal{I}_\varphi\right] \tag{32}$$

for every $N\in\mathbb{Z}_+$. At this point, we claim that $h_N \overset{\mathbb{P}\text{-a.s.}}{\longrightarrow} 0$ as $N\to\infty$. Let $\mathcal{E} := \{\omega\in\Omega : \lim_{n\to\infty} g_n(\omega) = g(\omega)\}\in\mathcal{F}$ and choose any $\omega\in\Omega$. Then for any given $\epsilon>0$, there is an $N(\omega,\epsilon)\in\mathbb{N}$ such that $|g_n(\omega)-g(\omega)|<\epsilon$ for all $n\geq N(\omega,\epsilon)$. This implies

$$h_{N(\omega,\epsilon)}(\omega) = \sup\{|g_n(\omega)-g(\omega)| : n\geq N(\omega,\epsilon)\}\leq\epsilon,$$

and therefore

$$\lim_{N\to\infty} h_N(\omega) \overset{\text{(b)}}{\leq} h_{N(\omega,\epsilon)}(\omega)\leq\epsilon, \tag{33}$$

where the step (b) follows from the fact that $\{h_N\}_{N=0}^{\infty}$ is a non-increasing sequence. As the inequality (33) holds for an arbitrarily chosen $\epsilon>0$, we obtain $\lim_{N\to\infty} h_N(\omega) = 0$ for all $\omega\in\mathcal{E}$. From the $\mathbb{P}$-almost sure convergence of the sequence $\{g_n\}_{n=0}^{\infty}$ to g, we know that $\mathbb{P}\{\mathcal{E}\} = 1$ and this completes the proof of the claim $h_N \overset{\mathbb{P}\text{-a.s.}}{\longrightarrow} 0$ as $N\to\infty$. Since $h_N\leq h_0$ for all $N\in\mathbb{Z}_+$ together with the fact $h_0\in L^1(\Omega,\mathcal{F},\mathbb{P})$, we may conclude from the dominated convergence theorem for conditional expectations that

$$\mathbb{E}\left[h_N|\mathcal{I}_\varphi\right] \overset{\mathbb{P}\text{-a.s.}}{\longrightarrow} 0 \tag{34}$$

as $N\to\infty$. Letting $N\to\infty$ in the bound (32), and then putting (34) into (32) yields

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} g_k\circ\varphi^k \overset{\mathbb{P}\text{-a.s.}}{\leq} \mathbb{E}\left[g|\mathcal{I}_\varphi\right],$$

as desired.

□

Applying Claim 1 to the sequence $\{-g_n\}_{n=0}^{\infty}$ gives

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g_k \circ \varphi^k \overset{\mathbb{P}\text{-a.s.}}{\geq} \mathbb{E}\left[g|\mathcal{I}_\varphi\right], \tag{35}$$

and combining this bound together with the bound (30) in Claim 1 completes the proof of the statement (i).

(ii) The L^1-convergence of $\{g_n\}_{n=0}^{\infty}$ to g implies $\lim_{n\to\infty} \mathbb{E}\left[|g_n - g|\right] = 0$. From the triangle inequality, one has

$$\begin{aligned} &\mathbb{E}\left[\left|\left(\frac{1}{n}\sum_{k=0}^{n-1} g_k \circ \varphi^k\right) - \mathbb{E}\left[g|\mathcal{I}_\varphi\right]\right|\right] \\ &\leq \underbrace{\mathbb{E}\left[\left|\left(\frac{1}{n}\sum_{k=0}^{n-1} g_k \circ \varphi^k\right) - \left(\frac{1}{n}\sum_{k=0}^{n-1} g \circ \varphi^k\right)\right|\right]}_{=:\ (\mathrm{Q1})} + \underbrace{\mathbb{E}\left[\left|\left(\frac{1}{n}\sum_{k=0}^{n-1} g \circ \varphi^k\right) - \mathbb{E}\left[g|\mathcal{I}_\varphi\right]\right|\right]}_{=:\ (\mathrm{Q2})}. \end{aligned} \tag{36}$$

We know that $\lim_{n\to\infty} (\mathrm{Q2}) = 0$ owing to the L^1-convergence part of the Birkhoff's ergodic theorem (*Theorem 6.2.1* in [1]). So, it suffices to prove $\lim_{n\to\infty} (\mathrm{Q1}) = 0$ by the bound (36). To this end, we observe that

$$\begin{aligned} (\mathrm{Q1}) &= \mathbb{E}\left[\frac{1}{n}\left|\sum_{k=0}^{n-1}\left(g_k \circ \varphi^k - g \circ \varphi^k\right)\right|\right] \\ &\overset{\text{(c)}}{\leq} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}\left[\left|g_k \circ \varphi^k - g \circ \varphi^k\right|\right] \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}\left[|g_k - g| \circ \varphi^k\right] \\ &\overset{\text{(d)}}{=} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}\left[|g_k - g|\right], \end{aligned} \tag{37}$$

where the step (c) is due to the triangle inequality, and the step (d) comes from Lemma 3. Since the standard convergence of a sequence in $\mathbb{R}$ implies the Cesàro convergence of the sequence, we can see that the last term of the inequality (37) converges to 0 as $n \to \infty$. Therefore, $\lim_{n\to\infty} (\mathrm{Q1}) = 0$ and thus

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\left(\frac{1}{n}\sum_{k=0}^{n-1} g_k \circ \varphi^k\right) - \mathbb{E}\left[g|\mathcal{I}_\varphi\right]\right|\right] = 0,$$

that is,

$$\frac{1}{n}\sum_{k=0}^{n-1} g_k \circ \varphi^k \xrightarrow{L^1} \mathbb{E}\left[g|\mathcal{I}_\varphi\right]$$

as $n \to \infty$.

Problem 6 (*Exercise 6.2.3.* in [1]: Wiener's maximal inequality).

Let $X' := X - \alpha$, $X'_k := X' \circ \varphi^k$ for $k \in \mathbb{Z}_+$, $S'_n := \sum_{k=0}^{n-1} X'_k$ for $n \in \mathbb{N}$ with $S'_0 := 0$, and

$$M'_n := \max\left\{S'_k : k \in [0:n]\right\} = \max\left\{0, S'_1, S'_2, \cdots, S'_n\right\}, \ \forall n \in \mathbb{N}.$$

Then for any $\omega \in \Omega$, we see that

$$
\begin{aligned}
D_k(\omega) > \alpha \Leftrightarrow A_j(\omega) = \frac{S_j(\omega)}{j} > \alpha \text{ for some } j \in [k] \\
\Leftrightarrow S_j(\omega) - \alpha j > 0 \text{ for some } j \in [k].
\end{aligned} \tag{38}
$$

Since

$$
S'_n(\omega) = \sum_{k=0}^{n-1} \left(X' \circ \varphi^k \right)(\omega) = \sum_{k=0}^{n-1} \left\{ \left(X \circ \varphi^k \right)(\omega) - \alpha \right\} = \sum_{k=0}^{n-1} \left(X \circ \varphi^k \right)(\omega) - \alpha n = S_n(\omega) - \alpha n,
$$

we obtain from (38) that

$$
\begin{aligned}
\{\omega \in \Omega : D_k(\omega) > \alpha\} &= \bigcup_{j=1}^{k} \{\omega \in \Omega : S_j(\omega) - \alpha j > 0\} \\
&= \bigcup_{j=1}^{k} \{\omega \in \Omega : S'_j(\omega) > 0\} \\
&= \left\{\omega \in \Omega : M'_k(\omega) > 0\right\}
\end{aligned} \tag{39}
$$

for every $k \in \mathbb{N}$. Since $X' \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, the maximal ergodic lemma (*Lemma 6.2.2* in [1]) implies

$$
\begin{aligned}
0 &\le \mathbb{E}\left[X' \cdot \mathbb{1}_{\{M'_k > 0\}} \right] \\
&= \mathbb{E}\left[(X - \alpha) \cdot \mathbb{1}_{\{M'_k > 0\}} \right] \\
&= \mathbb{E}\left[X \cdot \mathbb{1}_{\{M'_k > 0\}} \right] - \alpha \mathbb{P}\left\{ M'_k > 0 \right\} \\
&\overset{\text{(a)}}{=} \mathbb{E}\left[X \cdot \mathbb{1}_{\{M'_k > 0\}} \right] - \alpha \mathbb{P}\left\{ D_k > \alpha \right\} \\
&\le \mathbb{E}\left[|X| \cdot \mathbb{1}_{\{M'_k > 0\}} \right] - \alpha \mathbb{P}\left\{ D_k > \alpha \right\} \\
&\le \mathbb{E}\left[|X| \right] - \alpha \mathbb{P}\left\{ D_k > \alpha \right\},
\end{aligned}
$$

where the step (a) follows from the set relation (39), and thus we arrive at

$$
\mathbb{P}\left\{ D_k > \alpha \right\} \le \alpha^{-1} \mathbb{E}\left[|X| \right],
$$

as desired.

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.