MAS651 Theory of Stochastic Processes
Homework #5

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April 7, 2021

Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$. Also, we use the symbol $\mathbb{S}$ instead of S to denote the underlying state space of stochastic processes.

We assume throughout this homework that the underlying state space $\mathbb{S}$ is countable and it is equipped with the discrete σ-field $2^{\mathbb{S}}$ on $\mathbb{S}$. Since $(\mathbb{S}, 2^{\mathbb{S}})$ is a *nice measurable space*, it admits the canonical construction in *Section 5.2* in [1] of the probability measure $\mathbb{P}_\mu$ on the sequence space $(\Omega_0, \mathcal{F}_\infty)$ so that the sequence of coordinate maps $\{X_n(\omega) := \omega_n\}_{n=0}^\infty$ is a homogeneous Markov chain with initial distribution μ and transition probability $p(\cdot, \cdot) : \mathbb{S} \times 2^{\mathbb{S}} \to [0, 1]$. We remark that it is conventional to write $p(x, y) := p(x, \{y\})$ for $x, y \in \mathbb{S}$.

Problem 1 (*Exercise 5.6.1.* in [1]).

Observe that for any $n \in \mathbb{Z}_+$, we have $\mathbb{1}_{\{X_{n+1}=0\}} = \mathbb{1}_{\{X_1=0\}} \circ \theta_n$ on Ω_0. Thus,

$$\begin{aligned}
\mathbb{P}_\mu\{X_{n+1}=0\} &= \mathbb{E}_\mu\left[\mathbb{1}_{\{X_{n+1}=0\}}\right] \\
&= \mathbb{E}_\mu\left[\mathbb{1}_{\{X_1=0\}} \circ \theta_n\right] \\
&= \mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\mathbb{1}_{\{X_1=0\}} \circ \theta_n \middle| \mathcal{F}_n\right]\right] \\
&\overset{\text{(a)}}{=} \mathbb{E}_\mu\left[\mathbb{E}_{X_n}\left[\mathbb{1}_{\{X_1=0\}}\right]\right] \\
&= \mathbb{E}_\mu\left[\sum_{x\in\mathbb{S}} \mathbb{E}_{X_n}\left[\mathbb{1}_{\{X_1=0\}}\right]\mathbb{1}_{\{X_n=x\}}\right] \\
&= \mathbb{P}_0\{X_1=0\} \cdot \mathbb{P}_\mu\{X_n=0\} + \mathbb{P}_1\{X_1=0\} \cdot \mathbb{P}_\mu\{X_n=1\} \\
&= (1-\alpha)\mathbb{P}_\mu\{X_n=0\} + \beta\left(1-\mathbb{P}_\mu\{X_n=0\}\right) \\
&= \beta + (1-\alpha-\beta)\,\mathbb{P}_\mu\{X_n=0\},
\end{aligned} \tag{1}$$

where the step (a) holds by the Markov property (*Theorem 5.2.3* in [1]). Here, $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^\infty$ denotes the canonical filtration of the Markov chain $\{X_n\}_{n=0}^\infty$. From (1), we obtain

$$\mathbb{P}_\mu\{X_n=0\} - \frac{\beta}{\alpha+\beta} = (1-\alpha-\beta)\left(\mathbb{P}_\mu\{X_{n-1}=0\} - \frac{\beta}{\alpha+\beta}\right) \tag{2}$$

for all $n \in \mathbb{N}$. Now, we deduce our desired result via induction on n. The case $n = 0$ is trivial. Now, assume that we have

$$\mathbb{P}_\mu \{X_{k-1} = 0\} = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^{k-1}\left\{\mu(0) - \frac{\beta}{\alpha+\beta}\right\} \tag{3}$$

for $k \geq 1$. Putting the induction hypothesis (3) into the recursive relation (2) directly yields

$$\mathbb{P}_\mu \{X_k = 0\} = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^{k}\left\{\mu(0) - \frac{\beta}{\alpha+\beta}\right\},$$

which completes the proof.

Problem 2 (*Exercise 5.6.2.* in [1]).

Since $p(\cdot,\cdot) : \mathbb{S}\times\mathbb{S} \to [0,1]$ is irreducible, there is a positive integer $K(x,y) \in \mathbb{N}$ such that $p^{K(x,y)}(x,y) > 0$. Due to the aperiodicity of $p(\cdot,\cdot)$, we have $d_x = 1$ and so there is a positive integer $N(x) \in \mathbb{N}$ such that $p^n(x,x) > 0$ for all $n \geq N(x)$ by *Lemma 5.6.5* in [1]. Thus, one has

$$p^{n+N(x)+K(x,y)}(x,y) \overset{\text{(a)}}{\geq} p^{n+N(x)}(x,x)\cdot p^{K(x,y)}(x,y) > 0 \tag{4}$$

for all $n \in \mathbb{Z}_+$, where the step (a) follows from the Chapman-Kolmogorov equation. Let

$$\Gamma := \max\{N(x) + K(x,y) : (x,y) \in \mathbb{S}\times\mathbb{S}\},$$

which is finite since $\mathbb{S}\times\mathbb{S}$ is finite. Then for any $n \geq \Gamma$,

$$p^n(x,y) > 0,\ \forall (x,y) \in \mathbb{S}\times\mathbb{S},$$

since $n \geq \Gamma \geq N(x) + K(x,y)$ for all $(x,y) \in \mathbb{S}\times\mathbb{S}$ together with the fact (4). Hence, we have $p^n(x,y) > 0$ for all $(x,y) \in \mathbb{S}\times\mathbb{S}$, for any $n \geq \Gamma$, and it suffices to choose the desired integer m to be greater than Γ.

Remark 1. Let $(\mathbb{S},\mathcal{S})$ be a nice state space, and $(\Omega_0,\mathcal{F}_\infty)$ denote the sequence space obtained from $(\mathbb{S},\mathcal{S})$. For any probability measure $\mu : \mathcal{S} \to [0,1]$ on $(\mathbb{S},\mathcal{S})$, let $\mathbb{P}_\mu$ denote the canonical probability measure on the sequence space $(\Omega_0,\mathcal{F}_\infty)$ constructed in *Section 5.2* of [1] via Kolmogorov's extension theorem, and $\mathbb{P}_x := \mathbb{P}_{\delta_x}$ for $x \in \mathbb{S}$, where $\delta_x : \mathcal{S} \to [0,1]$ refers to the Dirac measure centered on the state $x \in \mathbb{S}$. Then, we know that

$$\mathbb{P}_\mu\{E\} = \int_{\mathbb{S}} \mu(\mathrm{d}x)\mathbb{P}_x\{E\} \tag{5}$$

for all $E \in \mathcal{F}_\infty$.

Problem 3 (*Exercise 5.6.3.* in [1]).

To begin with, we note that $\mathbb{S}^2 := \mathbb{S}\times\mathbb{S}$ is finite.

Claim 1. *The transition probability $\overline{p}(\cdot,\cdot) : \mathbb{S}^2\times\mathbb{S}^2 \to [0,1]$ on $\mathbb{S}^2$ defined by*

$$\overline{p}((x_1,y_1),(x_2,y_2)) := p(x_1,x_2)\cdot p(y_1,y_2),\ \forall (x_1,y_1),(x_2,y_2) \in \mathbb{S}^2,$$

is irreducible and aperiodic.

Proof of Claim 1.

We first claim that for all $n \in \mathbb{N}$,

$$\overline{p}^n\left((x_1, y_1), (x_2, y_2)\right) \geq p^n(x_1, x_2) \cdot p^n(y_1, y_2), \;\; \forall (x_1, y_1), (x_2, y_2) \in \mathbb{S}^2. \tag{6}$$

The proof of the claim (6) hinges upon the induction on n. The case $n = 1$ is immediate from the definition of $\overline{p}(\cdot, \cdot)$. Now, assume that (6) holds for the case $n = k - 1$, where $k \geq 2$. Then,

$$\begin{aligned}
\overline{p}^k\left((x_1, y_1), (x_2, y_2)\right) &\overset{\text{(a)}}{=} \sum_{(z,w) \in \mathbb{S}^2} \overline{p}\left((x_1, y_1), (z, w)\right) \overline{p}^{k-1}\left((z, w), (x_2, y_2)\right) \\
&\overset{\text{(b)}}{\geq} \sum_{(z,w) \in \mathbb{S}^2} p(x_1, z) p(y_1, w) \cdot p^{k-1}(z, x_2) p^{k-1}(w, y_2) \\
&= \left\{ \sum_{z \in \mathbb{S}} p(x_1, z) p^{k-1}(z, x_2) \right\} \left\{ \sum_{w \in \mathbb{S}} p(y_1, w) p^{k-1}(w, y_2) \right\} \\
&\overset{\text{(c)}}{=} p^k(x_1, x_2) p^k(y_1, y_2),
\end{aligned}$$

and this proves the claim (6). Here, the above steps (a)–(c) can be justified as follows:

(a) the Chapman-Kolmogorov equation;

(b) the induction hypothesis;

(c) the same reason as the step (a).

Due to the irreducibility of $p(\cdot, \cdot)$, there are positive integers $K, L \in \mathbb{N}$ such that

$$p^K(x_1, x_2) > 0 \quad \text{and} \quad p^L(y_1, y_2) > 0.$$

Also from the aperiodicity of $p(\cdot, \cdot)$, there is a positive integer $N_0 \in \mathbb{N}$ such that

$$p^n(x_1, x_1) > 0 \quad \text{and} \quad p^n(y_1, y_1) > 0$$

for all $n \geq N_0$. Thus, we have for every $n \geq N_0$,

$$\begin{aligned}
p^{L+n+K}(x_1, x_2) &\overset{\text{(d)}}{\geq} p^{L+n}(x_1, x_1) \cdot p^K(x_1, x_2) > 0; \\
p^{K+n+L}(y_1, y_2) &\overset{\text{(e)}}{\geq} p^{K+n}(y_1, y_1) \cdot p^L(y_1, y_2) > 0,
\end{aligned} \tag{7}$$

where the step (d) and (e) are consequences of the Chapman-Kolmogorov equation. So for all $n \geq K + L + N_0$, one has

$$\overline{p}^n\left((x_1, y_1), (x_2, y_2)\right) \overset{\text{(f)}}{\geq} p^n(x_1, x_2) \cdot p^n(y_1, y_2) \overset{\text{(g)}}{>} 0, \tag{8}$$

where the step (f) is due to Claim 1, and the step (g) is due to (7), and this establishes the irreducibility of $\overline{p}(\cdot, \cdot)$. Note that the integer $K + L + N_0$ depends on the choice of two states $(x_1, y_1), (x_2, y_2) \in \mathbb{S}^2$. Moreover, putting $(x_1, y_1) = (x_2, y_2) = (x, y) \in \mathbb{S}^2$ into the inequality (8) yields $\overline{p}^n\left((x, y), (x, y)\right) > 0$ for all but finitely many $n \in \mathbb{N}$. This implies $d_{(x,y)} = 1$ for all $(x, y) \in \mathbb{S}^2$, thereby $\overline{p}(\cdot, \cdot)$ is aperiodic.

□

Combining Claim 1 together with Problem 2 guarantees that there exists a positive integer $\Gamma \in \mathbb{N}$ such that

$$\overline{p}^{\Gamma}\left((x_1,y_1),(x_2,y_2)\right) > 0,\ \forall (x_1,y_1),(x_2,y_2) \in \mathbb{S}^2. \tag{9}$$

Let $\{Z_n := (X_n, Y_n)\}_{n=0}^{\infty}$ be the canonical homogeneous Markov chain constructed via the construction on the sequence space in *Section 5.2* of [1] with state space $\mathbb{S}^2$ and transition probability $\overline{p}(\cdot,\cdot) : \mathbb{S}^2 \times \mathbb{S}^2 \to [0,1]$. Further we let $\Delta := \left\{(x,x) \in \mathbb{S}^2 : x \in \mathbb{S}\right\}$ denote the *diagonal of* $\mathbb{S}^2$, and

$$\epsilon := \min\left\{\sum_{z\in\mathbb{S}} \overline{p}^{\Gamma}\left((x,y),(z,z)\right) : (x,y) \in \mathbb{S}^2\right\} = \min\left\{\mathbb{P}_{(x,y)}\left\{Z_\Gamma \in \Delta\right\} : (x,y) \in \mathbb{S}^2\right\} > 0.$$

At this point, recall that $T := \inf\{n \geq 1 : Z_n \in \Delta\}$. Then, we have

$$\begin{aligned}
\mathbb{P}_{(x,y)}\left\{T > \Gamma\right\} &= \mathbb{P}_{(x,y)}\left\{Z_1 \in \mathbb{S}^2 \setminus \Delta,\ Z_2 \in \mathbb{S}^2 \setminus \Delta, \cdots,\ Z_\Gamma \in \mathbb{S}^2 \setminus \Delta\right\} \\
&\leq \mathbb{P}_{(x,y)}\left\{Z_\Gamma \in \mathbb{S}^2 \setminus \Delta\right\} \\
&= 1 - \mathbb{P}_{(x,y)}\left\{Z_\Gamma \in \Delta\right\} \\
&\leq 1 - \epsilon
\end{aligned} \tag{10}$$

for all $(x,y) \in \mathbb{S}^2$. Thanks to Remark 1, we arrive at

$$\begin{aligned}
\mathbb{P}_{\nu}\left\{T > \Gamma\right\} &= \sum_{(x,y)\in\mathbb{S}^2} \nu(x,y) \cdot \mathbb{P}_{(x,y)}\left\{T > \Gamma\right\} \\
&\leq (1-\epsilon) \sum_{(x,y)\in\mathbb{S}^2} \nu(x,y) \\
&= 1 - \epsilon,
\end{aligned} \tag{11}$$

where $\nu(\cdot) : \mathbb{S}^2 \to [0,1]$ is any initial distribution $\{Z_n\}_{n=0}^{\infty}$. One can observe that for each $k \geq 2$, we have that if $T(\omega) > (k-1)\Gamma$,

$$\begin{aligned}
\mathbb{1}_{\{T>\Gamma\}} \circ \theta_{(k-1)\Gamma}(\omega) &= \begin{cases} 1 & \text{if } Z_1\left(\theta_{(k-1)\Gamma}(\omega)\right) \in \mathbb{S}^2 \setminus \Delta, \cdots, Z_\Gamma\left(\theta_{(k-1)\Gamma}(\omega)\right) \in \mathbb{S}^2 \setminus \Delta; \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } Z_{(k-1)\Gamma+1}(\omega) \in \mathbb{S}^2 \setminus \Delta, \cdots, Z_{k\Gamma}(\omega) \in \mathbb{S}^2 \setminus \Delta; \\ 0 & \text{otherwise} \end{cases} \\
&\overset{\text{(h)}}{=} \begin{cases} 1 & \text{if } Z_1(\omega) \in \mathbb{S}^2 \setminus \Delta, \cdots, Z_{k\Gamma}(\omega) \in \mathbb{S}^2 \setminus \Delta; \\ 0 & \text{otherwise} \end{cases} \\
&= \mathbb{1}_{\{T>k\Gamma\}}(\omega),
\end{aligned}$$

where the step (h) holds since $T(\omega) > (k-1)\Gamma$. In other words,

$$\mathbb{1}_{\{T>k\Gamma\}} = \left(\mathbb{1}_{\{T>\Gamma\}} \circ \theta_{(k-1)\Gamma}\right) \mathbb{1}_{\{T>(k-1)\Gamma\}} \tag{12}$$

on $(\mathbb{S}^2)^{\mathbb{Z}_+}$. Here, $\theta_n : (\mathbb{S}^2)^{\mathbb{Z}_+} \to (\mathbb{S}^2)^{\mathbb{Z}_+}$ denotes the shift operator on $(\mathbb{S}^2)^{\mathbb{Z}_+}$. Therefore,

$$\begin{aligned}
\mathbb{P}_\nu \{T > k\Gamma\} &= \mathbb{E}_\nu \left[\mathbb{1}_{\{T>k\Gamma\}}\right] \\
&\overset{\text{(i)}}{=} \mathbb{E}_\nu \left[\left(\mathbb{1}_{\{T>\Gamma\}} \circ \theta_{(k-1)\Gamma}\right) \mathbb{1}_{\{T>(k-1)\Gamma\}}\right] \\
&= \mathbb{E}_\nu \left[\mathbb{E}_\nu \left[\left(\mathbb{1}_{\{T>\Gamma\}} \circ \theta_{(k-1)\Gamma}\right) \mathbb{1}_{\{T>(k-1)\Gamma\}} \middle| \mathcal{F}^{\mathbf{Z}}_{(k-1)\Gamma}\right]\right] \\
&\overset{\text{(j)}}{=} \mathbb{E}_\nu \left[\mathbb{E}_\nu \left[\mathbb{1}_{\{T>\Gamma\}} \circ \theta_{(k-1)\Gamma} \middle| \mathcal{F}^{\mathbf{Z}}_{(k-1)\Gamma}\right] \mathbb{1}_{\{T>(k-1)\Gamma\}}\right] \\
&\overset{\text{(k)}}{=} \mathbb{E}_\nu \left[\mathbb{E}_{Z_{(k-1)\Gamma}} \left[\mathbb{1}_{\{T>\Gamma\}}\right] \mathbb{1}_{\{T>(k-1)\Gamma\}}\right] \\
&\overset{\text{(l)}}{\leq} (1-\epsilon)\mathbb{P}_\nu \{T > (k-1)\Gamma\},
\end{aligned} \tag{13}$$

where the above steps (i)–(l) can be validated as follows:

(i) the equality (12);

(j) $\{T > (k-1)\Gamma\} = (\mathbb{S}^2)^{\mathbb{Z}_+} \setminus \{T \leq (k-1)\Gamma\} \in \mathcal{F}^{\mathbf{Z}}_{(k-1)\Gamma}$, since T is a stopping time with respect to the canonical filtration $\{\mathcal{F}^{\mathbf{Z}}_n\}_{n=0}^\infty$ denotes the canonical filtration of the Markov chain $\{Z_n\}_{n=0}^\infty$, where

$$\mathcal{F}^{\mathbf{Z}}_n := \sigma(Z_0, Z_1, \cdots, Z_n) = \sigma((X_0, Y_0), (X_1, Y_1), \cdots, (X_n, Y_n))$$

for each $n \in \mathbb{Z}_+$;

(k) the Markov property (*Theorem 5.2.3* in [1]);

(l) the inequality (11).

So, we may inductively deduce from (13) that

$$\mathbb{P}_\nu \{T > k\Gamma\} \leq (1-\epsilon)^k \tag{14}$$

for every $k \in \mathbb{Z}_+$. Note that the bound (14) holds for any initial distribution $\nu(\cdot) : \mathbb{S}^2 \to [0, 1]$ of the Markov chain $\{Z_n\}_{n=0}^\infty$.

Finally, choose any $n \in \mathbb{Z}_+$ and let $k := \left\lfloor \frac{n}{\Gamma} \right\rfloor \in \mathbb{Z}_+$. Since $k\Gamma \leq n < (k+1)\Gamma$, we know $\frac{n}{\Gamma} - 1 < k \leq \frac{n}{\Gamma}$. Hence,

$$\begin{aligned}
\mathbb{P}_\nu \{T > n\} &\leq \mathbb{P}_\nu \{T > k\Gamma\} \\
&\overset{\text{(m)}}{\leq} (1-\epsilon)^k \\
&\overset{\text{(n)}}{\leq} (1-\epsilon)^{\frac{n}{\Gamma}-1} \\
&= \frac{1}{1-\epsilon} \cdot \left\{(1-\epsilon)^{\frac{1}{\Gamma}}\right\}^n,
\end{aligned} \tag{15}$$

where the step (m) makes use of the bound (14), and the step (n) holds since $0 < 1-\epsilon < 1$ and $k > \frac{n}{\Gamma} - 1$. By letting $C := \frac{1}{1-\epsilon} \in (0, +\infty)$ and $r := (1-\epsilon)^{\frac{1}{\Gamma}} \in (0, 1)$, the bound (15) establishes the desired result.

Problem 4 (*Exercise 5.6.5.* in [1]: Strong law for additive functionals)**.**

(i) We first prove the following useful result inspired by *Exercise 5.3.1* in [1]:

Lemma 1. *Let $\{X_n\}_{n=0}^{\infty}$ be a homogeneous Markov chain with countable state space $\mathbb{S}$ and transition probability $p(\cdot,\cdot) : \mathbb{S}\times\mathbb{S} \to [0,1]$, and $x \in \mathbb{S}$ be a recurrent state of the chain. For $k \in \mathbb{N}$, let $\tau_k := T_x^k - T_x^{k-1}$ be the k-th inter-arrival time to state x, and $V_k := \left(\tau_k, X_{T_x^{k-1}}, X_{T_x^{k-1}+1}, \cdots, X_{T_x^k-1}\right)$. Then given any probability distribution $\mu(\cdot) : \mathbb{S} \to [0,1]$, the sequence of random vectors $\{V_k : k \geq 2\}$ are independent and identically distributed under the canonical probability measure $\mathbb{P}_\mu$ defined on the sequence space $(\Omega_0, \mathcal{F}_\infty)$, conditionally on the event $\{T_x < +\infty\}$, where $T_x := T_x^1$ is the first hitting time to state x.*

Proof of Lemma 1.

To begin with, we may observe that if $T_x^{k-1}(\omega) < +\infty$, then $\mathbb{1}_{\{V_1=v\}} \circ \theta_{T_x^{k-1}} = \mathbb{1}_{\{V_k=v\}}$ for all $k \geq 2$ and $v \in \mathbb{V} := \bigcup_{n=1}^{\infty} (\{n\} \times \mathbb{S}^n)$. That is,

$$\mathbb{1}_{\{V_k=v\}} = \left(\mathbb{1}_{\{V_1=v\}} \circ \theta_{T_x^{k-1}}\right) \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}} \tag{16}$$

on Ω_0, since $\{V_k = v\} \subseteq \left\{T_x^k < +\infty\right\} \subseteq \left\{T_x^{k-1} < +\infty\right\}$. Thus, for every $k \geq 2$, one has

$$\begin{aligned}
\mathbb{P}_\mu\left\{ V_k = v \middle| \mathcal{F}_{T_x^{k-1}}\right\} &= \mathbb{E}_\mu\left[\mathbb{1}_{\{V_k=v\}} \middle| \mathcal{F}_{T_x^{k-1}}\right] \\
&\overset{\text{(a)}}{=} \mathbb{E}_\mu\left[\left(\mathbb{1}_{\{V_1=v\}} \circ \theta_{T_x^{k-1}}\right) \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}} \middle| \mathcal{F}_{T_x^{k-1}}\right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{V_1=v\}} \circ \theta_{T_x^{k-1}} \middle| \mathcal{F}_{T_x^{k-1}}\right] \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}} \\
&\overset{\text{(c)}}{=} \mathbb{E}_{X_{T_x^{k-1}}}\left[\mathbb{1}_{\{V_1=v\}}\right] \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}} \\
&\overset{\text{(d)}}{=} \mathbb{E}_x\left[\mathbb{1}_{\{V_1=v\}}\right] \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}} \\
&= \mathbb{P}_x\left\{V_1 = v\right\} \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}}
\end{aligned} \tag{17}$$

$\mathbb{P}_\mu$-almost surely, where the above steps (a)–(d) can be justified as follows:

(a) the equality (16);

(b) $\left\{T_x^{k-1} < +\infty\right\} \in \mathcal{F}_{T_x^{k-1}}$, since

$$\left\{T_x^{k-1} < +\infty\right\} \cap \left\{T_x^{k-1} = n\right\} = \left\{T_x^{k-1} = n\right\} \in \mathcal{F}_{T_x^{k-1}}$$

for every $n \in \mathbb{Z}_+$;

(c) the strong Markov property (*Theorem 5.2.5* in [1]);

(d) if $T_x^{k-1} < +\infty$, then $X_{T_x^{k-1}} = x$ for $k \geq 2$.

One can immediately deduce from (17) that

$$\mathbb{P}_\mu\left\{V_k = v\right\} = \mathbb{E}_\mu\left[\mathbb{P}_\mu\left\{ V_k = v \middle| \mathcal{F}_{T_x^{k-1}}\right\}\right] = \mathbb{P}_x\left\{V_1 = v\right\} \mathbb{P}_\mu\left\{T_x^{k-1} < +\infty\right\}. \tag{18}$$

At this point, we claim that for all $k \in \mathbb{N}$, $\mathbb{P}_\mu\left\{T_x^k < +\infty\right\} = \mathbb{P}_\mu\left\{T_x < +\infty\right\}$. If $k = 1$, there's nothing to prove and we may assume that $k \geq 2$. One can easily see that if $T_x^{k-1}(\omega) < +\infty$, then

$$\left(\mathbb{1}_{\{T_x<+\infty\}} \circ \theta_{T_x^{k-1}}\right)(\omega) = \begin{cases} 1 & \text{if } X_n\left(\theta_{T_x^{k-1}}(\omega)\right) = x \text{ for some } n > 0; \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } X_n(\omega) = x \text{ for some } n > T_x^{k-1}(\omega); \\ 0 & \text{otherwise} \end{cases}$$
$$= \mathbb{1}_{\left\{T_x^k<+\infty\right\}}(\omega).$$

More succinctly, we have

$$\mathbb{1}_{\left\{T_x^k<+\infty\right\}} = \left(\mathbb{1}_{\{T_x<+\infty\}} \circ \theta_{T_x^{k-1}}\right) \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}} \tag{19}$$

on Ω_0. Hence,

$$\begin{aligned}
\mathbb{P}_\mu\left\{T_x^k < +\infty\right\} &= \mathbb{E}_\mu\left[\mathbb{1}_{\left\{T_x^k<+\infty\right\}}\right] \\
&\overset{\text{(e)}}{=} \mathbb{E}_\mu\left[\left(\mathbb{1}_{\{T_x<+\infty\}} \circ \theta_{T_x^{k-1}}\right) \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}}\right] \\
&= \mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\left(\mathbb{1}_{\{T_x<+\infty\}} \circ \theta_{T_x^{k-1}}\right) \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}}\middle| \mathcal{F}_{T_x^{k-1}}\right]\right] \\
&\overset{\text{(f)}}{=} \left[\mathbb{E}_\mu\left[\mathbb{1}_{\{T_x<+\infty\}} \circ \theta_{T_x^{k-1}}\middle| \mathcal{F}_{T_x^{k-1}}\right] \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}}\right] \\
&\overset{\text{(g)}}{=} \mathbb{E}_\mu\left[\mathbb{E}_{X_{T_x^{k-1}}}\left[\mathbb{1}_{\{T_x<+\infty\}}\right] \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}}\right] \\
&\overset{\text{(h)}}{=} \mathbb{E}_\mu\left[\mathbb{E}_x\left[\mathbb{1}_{\{T_x<+\infty\}}\right] \mathbb{1}_{\left\{T_x^{k-1}<+\infty\right\}}\right] \\
&= \rho_{xx} \cdot \mathbb{P}_\mu\left\{T_x^{k-1} < +\infty\right\} \\
&\overset{\text{(i)}}{=} \mathbb{P}_\mu\left\{T_x^{k-1} < +\infty\right\},
\end{aligned} \tag{20}$$

where the above steps (e)–(i) hold since:

(e) the equality (19);

(f) the same reason as the step (b);

(g) the same reason as the step (c);

(h) the same reason as the step (d);

(i) $\rho_{xx} = 1$, because the state $x \in \mathbb{S}$ is recurrent,

thereby one can deduce our desired claim from (20) inductively. So, the equation (18) becomes

$$\mathbb{P}_x\{V_1 = v\} \mathbb{P}_\mu\{T_x < +\infty\} = \mathbb{P}_\mu\{V_k = v\} \overset{\text{(j)}}{=} \mathbb{P}_\mu\{V_k = v, T_x < +\infty\},$$

where the step (j) holds since $\{V_k = v\} \subseteq \{T_x < +\infty\}$, thereby we arrive at

$$\mathbb{P}_x\{V_1 = v\} = \mathbb{P}_\mu\{V_k = v | T_x < +\infty\} \tag{21}$$

for all $k \geq 2$. Thus, $\{V_k : k \geq 2\}$ are identically distributed under the canonical probability measure $\mathbb{P}_\mu$ on the sequence space $(\Omega_0, \mathcal{F}_\infty)$, conditionally on the event $\{T_x < +\infty\}$.

As the final step, it remains to establish the conditional independence of the sequence $\{V_k : k \geq 2\}$ given the event $\{T_x < +\infty\}$. Let $\{v_k : k \geq 2\}$ be any sequence in $\mathbb{V}$. Since $\mathbb{P}_\mu \{T_x < +\infty\} = \mathbb{P}_\mu \left\{T_x^k < +\infty\right\}$ for all $k \in \mathbb{N}$, we know that $\mathbb{1}_{\mathbb{P}_\mu\{T_x<+\infty\}} \overset{\mathbb{P}_\mu\text{-a.s.}}{=} \mathbb{1}_{\mathbb{P}_\mu\{T_x^k<+\infty\}}$ for all $k \in \mathbb{N}$. Let us begin our argument from the equation (17): for every $k \geq 2$,

$$\mathbb{P}_\mu \left\{ V_k = v_k \middle| \mathcal{F}_{T_x^{k-1}} \right\} \overset{\mathbb{P}_\mu\text{-a.s.}}{=} \mathbb{P}_x \{V_1 = v_k\} \mathbb{1}_{\{T_x<+\infty\}}. \tag{22}$$

It is easy to see that $\{V_2 = v_2, V_3 = v_3, \cdots, V_{k-1} = v_{k-1}\} \in \mathcal{F}_{T_x^{k-1}}$, so

$$\begin{aligned}
&\mathbb{P}_\mu \{V_2 = v_2, V_3 = v_3, \cdots, V_k = v_k\} \\
&= \int_{\{V_2=v_2,V_3=v_3,\cdots,V_{k-1}=v_{k-1}\}} \mathbb{1}_{\{V_k=v_k\}} \mathrm{d}\mathbb{P}_\mu \\
&\overset{(\mathrm{k})}{=} \int_{\{V_2=v_2,V_3=v_3,\cdots,V_{k-1}=v_{k-1}\}} \mathbb{P}_x \{V_1 = v_k\} \mathbb{1}_{\{T_x<+\infty\}} \mathrm{d}\mathbb{P}_\mu \\
&= \mathbb{P}_x \{V_1 = v_k\} \mathbb{P}_\mu \{V_2 = v_2, V_3 = v_3, \cdots, V_{k-1} = v_{k-1}, T_x < +\infty\} \\
&\overset{(\mathrm{l})}{=} \mathbb{P}_\mu \{ V_k = v_k | T_x < +\infty\} \mathbb{P}_\mu \{ V_2 = v_2, V_3 = v_3, \cdots, V_{k-1} = v_{k-1} | T_x < +\infty\} \mathbb{P}_\mu \{T_x < +\infty\},
\end{aligned} \tag{23}$$

where the step (k) comes from (22), and the step (l) is due to the equation (21). Hence, we reach

$$\begin{aligned}
&\mathbb{P}_\mu \{ V_2 = v_2, V_3 = v_3, \cdots, V_{k-1} = v_{k-1} | T_x < +\infty\} \mathbb{P}_\mu \{ V_k = v_k | T_x < +\infty\} \\
&= \frac{\mathbb{P}_\mu \{V_2 = v_2, V_3 = v_3, \cdots, V_k = v_k\}}{\mathbb{P}_\mu \{T_x < +\infty\}} \\
&\overset{(\mathrm{m})}{=} \frac{\mathbb{P}_\mu \{V_2 = v_2, V_3 = v_3, \cdots, V_k = v_k, T_x < +\infty\}}{\mathbb{P}_\mu \{T_x < +\infty\}} \\
&= \mathbb{P}_\mu \{ V_2 = v_2, V_3 = v_3, \cdots, V_k = v_k | T_x < +\infty\},
\end{aligned} \tag{24}$$

where the step (m) holds since $\{V_2 = v_2, V_3 = v_3, \cdots, V_k = v_k\} \subseteq \{T_x < +\infty\}$. Therefore, we may inductively conclude that

$$\mathbb{P}_\mu \{ V_2 = v_2, V_3 = v_3, \cdots, V_n = v_n | T_x < +\infty\} = \prod_{k=2}^{n} \mathbb{P}_\mu \{ V_k = v_k | T_x < +\infty\}$$

for all $n \geq 2$ and $v_2, v_3, \cdots, v_n \in \mathbb{V}$. Hence, $\{V_k : k \geq 2\}$ is a sequence of independent and identically distributed random vectors under $\mathbb{P}_\mu$ defined on $(\Omega_0, \mathcal{F}_\infty)$, conditionally on the event $\{T_x < +\infty\}$.

□

Now, it's time to complete the proof of the statement (i). Since $p(\cdot, \cdot)$ is irreducible and has a stationary distribution $\pi(\cdot) : \mathbb{S} \to [0, 1]$, it is positive recurrent by *Theorem 5.5.12* in [1]. Thanks to Lemma 1, for any state $x \in \mathbb{S}$, the sequence of random vectors,

$$\left\{ V_k := \left(\tau_k, X_{T_x^{k-1}}, X_{T_x^{k-1}+1}, \cdots, X_{T_x^k - 1} \right) : k \geq 2 \right\},$$

are independent and identically distributed under $\mathbb{P}_\mu$, conditionally on the event $\{T_x < +\infty\}$. Here, we may observe that for any event $E \in \mathcal{F}_\infty$,

$$\begin{aligned}
\mathbb{P}_\mu \{E\} &= \mathbb{P}_\mu \{ E | T_x < +\infty\} \mathbb{P}_\mu \{T_x < +\infty\} + \mathbb{P}_\mu \{ E | T_x = +\infty\} \mathbb{P}_\mu \{T_x = +\infty\} \\
&\overset{(\mathrm{n})}{=} \mathbb{P}_\mu \{ E | T_x < +\infty\},
\end{aligned} \tag{25}$$

where the step (n) can be justified as follows: since $p(\cdot,\cdot)$ is irreducible and the state $x \in \mathbb{S}$ is recurrent, $\rho_{yx} = 1$ for all $y \in \mathbb{S}$ due to *Theorem 5.3.2* in [1]. So, Remark 1 implies

$$\mathbb{P}_\mu \{T_x < +\infty\} = \sum_{y \in \mathbb{S}} \mu(y) \cdot \mathbb{P}_y \{T_x < +\infty\} = \sum_{y \in \mathbb{S}} \mu(y) \cdot \rho_{yx} = 1,$$

as desired. From the observation (25), one can see from Lemma 1 that $\{V_k\}_{k=2}^{\infty}$ is a sequence of independent and identically distributed random vectors under the canonical probability measure $\mathbb{P}_\mu$ defined on the sequence space $(\Omega_0, \mathcal{F}_\infty)$, not necessarily conditionally on the event $\{T_x < +\infty\}$. Furthermore, we obtain from (21) together with the observation (25) that

$$\mathbb{P}_x \{V_1 = v\} = \mathbb{P}_\mu \{V_k = v\} \tag{26}$$

for all $k \geq 2$ and $v \in \mathbb{V}$. Now, define the function $F : \mathbb{V} \to \mathbb{R}$ by

$$F(n, x_0, x_1, \cdots, x_{n-1}) := \sum_{j=0}^{n-1} f(x_j), \ \forall (n, x_0, x_1, \cdots, x_{n-1}) \in \mathbb{V}.$$

Since $V_k^f = F(V_{k+1})$ for $k \in \mathbb{N}$, $\left\{V_k^f\right\}_{k=1}^{\infty}$ is a sequence of independent and identically distributed random variables under $\mathbb{P}_\mu$, for any initial distribution $\mu(\cdot) : \mathbb{S} \to [0,1]$ of the Markov chain.

It still remains to prove the $\mathbb{P}_\pi$-integrability of V_k^f for $k \geq 2$, *i.e.*, $\mathbb{E}_\pi\left[\left|V_k^f\right|\right] < +\infty$ for all $k \geq 2$. To begin with, we introduce the following conventions:

$$\pi(f) := \sum_{y \in \mathbb{S}} f(y)\pi(y) \quad \text{and} \quad \pi(|f|) := \sum_{y \in \mathbb{S}} |f(y)|\,\pi(y).$$

Then, the following bound holds:

$$
\begin{aligned}
\mathbb{E}_\pi\left[\left|V_k^f\right|\right] &= \mathbb{E}_\pi\left[\left|\sum_{j=T_x^k}^{T_x^{k+1}-1} f(X_j)\right|\right] \\
&\leq \mathbb{E}_\pi\left[\sum_{j=T_x^k}^{T_x^{k+1}-1} |f(X_j)|\right] \\
&\stackrel{\text{(o)}}{=} \sum_{1\leq a<b<+\infty} \mathbb{E}_\pi\left[\sum_{j=T_x^k}^{T_x^{k+1}-1} |f(X_j)| \,\middle|\, T_x^k = a, T_x^{k+1} = b\right] \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\
&= \sum_{1\leq a<b<+\infty} \mathbb{E}_\pi\left[\sum_{j=a}^{b-1} |f(X_j)|\right] \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\
&= \sum_{1\leq a<b<+\infty} \left[\sum_{j=a}^{b-1} \underbrace{\mathbb{E}_\pi\left[|f(X_j)|\right]}_{=\ \pi(|f|)}\right] \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\
&\stackrel{\text{(p)}}{=} \pi\left(|f|\right) \sum_{j=a}^{b-1} (b-a) \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\
&= \pi\left(|f|\right) \sum_{n=1}^{\infty} n \cdot \mathbb{P}_\pi\left\{\tau_{k+1} = n\right\} \\
&\stackrel{\text{(q)}}{=} \pi\left(|f|\right) \sum_{n=1}^{\infty} n \cdot \mathbb{P}_x\left\{T_x = n\right\} \\
&= \pi\left(|f|\right) \mathbb{E}_x\left[T_x\right] \stackrel{\text{(r)}}{<} +\infty,
\end{aligned}
$$

where the above steps (o)–(r) can be validated as follows:

(o) we have $\mathbb{P}_\pi\left\{T_x^k < +\infty\right\} = 1$ for all $n \in \mathbb{N}$ due to Remark 1;

(p) $\pi(\cdot) : \mathbb{S} \to [0, 1]$ is a stationary distribution for $p(\cdot, \cdot)$;

(q) from (26), we get

$$\mathbb{P}_x\left\{T_x = n\right\} = \mathbb{P}_\mu\left\{\tau_k = n\right\}$$

for all $n \in \mathbb{N}$, $k \geq 2$, and any initial distribution $\mu(\cdot)$ of the Markov chain;

(r) the state $x \in \mathbb{S}$ is positive recurrent,

and this completes the proof of the statement (i).

(ii) Firstly, one can see that

$$\begin{aligned} K_n &:= \inf \left\{ k \in \mathbb{Z}_+ : T_x^k \geq n \right\} \\ &= \inf \left\{ k \in \mathbb{Z}_+ : T_x^k > n-1 \right\} \\ &= \inf \left\{ k \in \mathbb{Z}_+ : \sum_{j=1}^{n-1} \mathbb{1}_{\{X_j = x\}} \leq k-1 \right\} \\ &= \sum_{j=1}^{n-1} \mathbb{1}_{\{X_j = x\}} + 1 \\ &= N_{n-1}(x) + 1. \end{aligned} \tag{27}$$

We defined the events

$$\mathcal{E}_1 := \left\{ \omega \in \Omega_0 : \lim_{n\to\infty} \frac{K_n(\omega)}{n} = \frac{1}{\mathbb{E}_x[T_x]} \right\} \in \mathcal{F}_\infty;$$
$$\mathcal{E}_2 := \left\{ \omega \in \Omega_0 : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} V_k^f(\omega) = \mathbb{E}_\pi \left[V_1^f \right] \right\} \in \mathcal{F}_\infty.$$

Due to *Theorem 5.6.1* in [1], we know that for any state $y \in \mathbb{S}$,

$$\frac{N_n(x)}{n} \overset{n\to\infty}{\longrightarrow} \frac{1}{\mathbb{E}_x[T_x]} \mathbb{1}_{\{T_x < +\infty\}} \overset{\text{(r)}}{=} \frac{1}{\mathbb{E}_x[T_x]}$$

$\mathbb{P}_y$-almost surely, where the step (r) follows from the fact $\mathbb{P}_y\{T_x < +\infty\} = \rho_{yx} = 1$ for every $y \in \mathbb{S}$, which holds by *Theorem 5.3.2* in [1]. Thus,

$$\frac{K_n}{n} \overset{\text{(s)}}{=} \frac{N_{n-1}}{n-1}\left(1 - \frac{1}{n}\right) + \frac{1}{n} \overset{n\to\infty}{\longrightarrow} \frac{1}{\mathbb{E}_x[T_x]}$$

$\mathbb{P}_y$-almost surely, where the step (s) makes use of the observation (27). So, we get $\mathbb{P}_y\{\mathcal{E}_1\} = 1$ for all $y \in \mathbb{S}$.

On the other hand, we know from the statement (i) that $\mathbb{E}_\pi\left[\left|V_k^f\right|\right] < +\infty$ for all $k \geq 2$. The strong law of large number yields $\mathbb{P}_\pi\{\mathcal{E}_2\} = 1$. Since the transition probability $p(\cdot,\cdot)$ is irreducible, there is a positive integer $K(x,y) \in \mathbb{N}$ such that $p^{K(x,y)}(x,y) > 0$ for every $(x,y) \in \mathbb{S} \times \mathbb{S}$. If there exists a state $z \in \mathbb{S}$ such that $\pi(z) = 0$, then

$$0 = \pi(z) = \sum_{x\in\mathbb{S}} \pi(x) \cdot p^{K(y,z)}(x,z) \geq \pi(y) \cdot p^{K(y,z)}(y,z),$$

which implies $\pi(y) = 0$ for all $y \in \mathbb{S}$. This violates the fact that $\pi(\cdot)$ has total mass 1, *i.e.*, $\sum_{x\in\mathbb{S}} \pi(x) = 1$, and therefore we find that $\pi(x) > 0$ for all $x \in \mathbb{S}$. Remark 1 implies

$$1 = \mathbb{P}_\pi\{\mathcal{E}_2\} = \sum_{y\in\mathbb{S}} \pi(y) \mathbb{P}_y\{\mathcal{E}_2\},$$

and this yields $\mathbb{P}_y\{\mathcal{E}_2\} = 1$ for all $y \in \mathbb{S}$, since $\pi(y) > 0$ for all $y \in \mathbb{S}$. Hence, we arrive at $\mathbb{P}_y\{\mathcal{E}_1 \cap \mathcal{E}_2\} = 1$ for all $y \in \mathbb{S}$, and employing Remark 1 again yields

$$\mathbb{P}_\mu\{\mathcal{E}_1 \cap \mathcal{E}_2\} = \sum_{y\in\mathbb{S}} \mu(y) \mathbb{P}_y\{\mathcal{E}_1 \cap \mathcal{E}_2\} = \sum_{y\in\mathbb{S}} \mu(y) = 1$$

for any initial distribution $\mu(\cdot) : \mathbb{S} \to [0,1]$ of the Markov chain.

Finally, for every $\omega \in \mathcal{E}_1 \cap \mathcal{E}_2$, one can see that $\lim_{n\to\infty} K_n(\omega) = +\infty$ since $\mathbb{E}_x[T_x] < +\infty$. Thus,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{K_n(\omega)} V_k^f(\omega) &= \lim_{n\to\infty} \frac{K_n(\omega)}{n} \cdot \frac{1}{K_n(\omega)} \sum_{k=1}^{K_n(\omega)} V_k^f(\omega) \\ &= \frac{\mathbb{E}_\pi\left[V_1^f\right]}{\mathbb{E}_x[T_x]} \\ &\overset{\text{(t)}}{=} \pi(f),\end{aligned}$$

where the step (t) can be verified as follows:

$$\begin{aligned}\mathbb{E}_\pi\left[V_1^f\right] &= \mathbb{E}_\pi\left[\sum_{j=T_x^k}^{T_x^{k+1}-1} f(X_j)\right] \\ &\overset{\text{(u)}}{=} \sum_{1\le a<b<+\infty} \mathbb{E}_\pi\left[\sum_{j=T_x^k}^{T_x^{k+1}-1} f(X_j) \,\middle|\, T_x^k = a, T_x^{k+1} = b\right] \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\ &= \sum_{1\le a<b<+\infty} \mathbb{E}_\pi\left[\sum_{j=a}^{b-1} f(X_j)\right] \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\ &= \sum_{1\le a<b<+\infty} \left[\sum_{j=a}^{b-1} \underbrace{\mathbb{E}_\pi[f(X_j)]}_{=\,\pi(f)}\right] \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\ &\overset{\text{(v)}}{=} \pi(f) \sum_{j=a}^{b-1} (b-a) \mathbb{P}_\pi\left\{T_x^k = a, T_x^{k+1} = b\right\} \\ &= \pi(f) \sum_{n=1}^{\infty} n \cdot \mathbb{P}_\pi\{\tau_{k+1} = n\} \\ &\overset{\text{(w)}}{=} \pi(f) \sum_{n=1}^{\infty} n \cdot \mathbb{P}_x\{T_x = n\} \\ &= \pi(f)\mathbb{E}_x[T_x],\end{aligned}$$

where the steps (u), (v), and (w) hold by the same reason as the steps (o), (p), and (q), respectively. So,

$$\frac{1}{n} \sum_{m=1}^{K_n} V_m^f \overset{n\to\infty}{\longrightarrow} \frac{\mathbb{E}_\pi\left[V_1^f\right]}{\mathbb{E}_x[T_x]} = \pi(f)$$

$\mathbb{P}_\mu$-almost surely, for any initial distribution $\mu(\cdot)$ of the Markov chain.

(iii) To begin with, we provide the following critical lemma:

Lemma 2. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[|X_1|] < +\infty$. Then,*

$$\frac{1}{n} \max\{|X_k| : k \in [n]\} \overset{\mathbb{P}\text{-a.s.}}{\longrightarrow} 0 \tag{28}$$

as $n \to \infty$.

Proof of Lemma 2.

Firstly, we fix any $\epsilon > 0$. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}\{|X_n| \geq n\epsilon\} &\stackrel{\text{(x)}}{=} \sum_{n=1}^{\infty} \mathbb{P}\{|X_1| \geq n\epsilon\} \\
&= \sum_{n=1}^{\infty} \int_{n-1}^{n} \mathbb{P}\left\{\frac{|X_1|}{\epsilon} \geq n\right\} \mathrm{d}t \\
&\stackrel{\text{(y)}}{\leq} \sum_{n=1}^{\infty} \int_{n-1}^{n} \mathbb{P}\left\{\frac{|X_1|}{\epsilon} \geq t\right\} \mathrm{d}t \\
&= \int_{0}^{\infty} \mathbb{P}\left\{\frac{|X_1|}{\epsilon} \geq t\right\} \mathrm{d}t \\
&\stackrel{\text{(z)}}{=} \frac{\mathbb{E}[|X_1|]}{\epsilon} < +\infty,
\end{aligned} \tag{29}$$

where the above steps (x)–(z) can be verified as follows:

(x) the sequence $\{X_n\}_{n=1}^{\infty}$ are identically distributed;

(y) for $n-1 \leq t \leq n$, we have $\mathbb{P}\left\{\frac{|X_1|}{\epsilon} \geq n\right\} \leq \mathbb{P}\left\{\frac{|X_1|}{\epsilon} \geq t\right\}$;

(z) *Lemma 2.2.13* in [1].

From the observation (29), we employ the first Borel-Cantelli lemma:

$$\mathbb{P}\left\{\limsup_{n\to\infty} \{|X_n| \geq n\epsilon\}\right\} = 0.$$

So, we conclude that for any $\epsilon > 0$,

$$\mathbb{P}\left\{\liminf_{n\to\infty} \{|X_n| < n\epsilon\}\right\} = 1. \tag{30}$$

Now, let $A_k := \liminf_{n\to\infty} \left\{\frac{|X_n|}{n} < \frac{1}{k}\right\}$ for $k \in \mathbb{N}$, and $A := \bigcap_{k=1}^{\infty} A_k$. From (30), we know $\mathbb{P}\{A\} = 1$ and if $\omega \in A$, then $\frac{|X_n(\omega)|}{n} < \frac{1}{k}$ for all but finitely many $n \in \mathbb{N}$. Thus,

$$\limsup_{n\to\infty} \frac{|X_n(\omega)|}{n} \leq \frac{1}{k}$$

for all $k \in \mathbb{N}$, and letting $k \to \infty$ yields the desired result.

□

By replacing f by $|f|$ in the statement (i), one can see that $\left\{V_k^{|f|} : k \in \mathbb{N}\right\}$ is a sequence of independent and identically distributed random variables under the canonical probability measure $\mathbb{P}_\mu$ on the sequence space $(\Omega_0, \mathcal{F}_\infty)$, for any initial distribution $\mu(\cdot)$ of the Markov chain, and $\mathbb{E}_\pi\left[V_k^{|f|}\right] < +\infty$ for all $k \in \mathbb{N}$. Applying Lemma 2 gives

$$\frac{1}{n} \max\left\{V_k^{|f|} : k \in [n]\right\} \stackrel{n\to\infty}{\longrightarrow} 0 \tag{31}$$

$\mathbb{P}_\pi$-almost surely. Let

$$\begin{aligned}
\mathcal{E}_3 &:= \left\{\omega \in \Omega_0 : \lim_{n\to\infty} \frac{1}{n} \max\left\{V_k^{|f|}(\omega) : k \in [n]\right\} = 0\right\} \in \mathcal{F}_\infty; \\
\mathcal{E}_4 &:= \left\{\omega \in \Omega_0 : T_x^k(\omega) < +\infty \text{ for all } k \in \mathbb{N}\right\} \in \mathcal{F}_\infty.
\end{aligned}$$

Then, (31) implies $\mathbb{P}_\pi\{\mathcal{E}_3\} = 1 = \sum_{y\in\mathbb{S}} \pi(y)\cdot\mathbb{P}_y\{\mathcal{E}_3\}$. Since $\pi(y) > 0$ for all $y \in \mathbb{S}$, we have $\mathbb{P}_y\{\mathcal{E}_3\} = 1$ for all $y \in \mathbb{S}$. Hence, Remark 1 yields

$$\mathbb{P}_\mu\{\mathcal{E}_3\} = \sum_{y\in\mathbb{S}} \mu(y)\cdot\mathbb{P}_y\{\mathcal{E}_3\} = \sum_{y\in\mathbb{S}} \mu(y) = 1,$$

and this establishes $\frac{1}{n}\max\left\{V_k^{|f|} : k \in [n]\right\} \overset{n\to\infty}{\longrightarrow} 0$, $\mathbb{P}_\mu$-almost surely, for any initial distribution $\mu(\cdot)$ of the Markov chain.

Furthermore, the irreducibility of $p(\cdot,\cdot)$ together with the recurrence of the state $x \in \mathbb{S}$ yields $\mathbb{P}_y\{\mathcal{E}_4\} = 1$ for all $y \in \mathbb{S}$. Therefore, we arrive at $\mathbb{P}_y\{\mathcal{E}_3 \cap \mathcal{E}_4\} = 1$ for all $y \in \mathbb{S}$, thereby from Remark 1,

$$\mathbb{P}_\mu\{\mathcal{E}_3\cap\mathcal{E}_4\} = \sum_{y\in\mathbb{S}} \mu(y)\cdot\mathbb{P}_y\{\mathcal{E}_3\cap\mathcal{E}_4\} = \sum_{y\in\mathbb{S}} \mu(y) = 1$$

for any initial distribution $\mu(\cdot)$ of the Markov chain. Since we know that $\mathbb{P}_\mu\{\mathcal{E}_1\cap\mathcal{E}_2\}$, we finally obtain

$$\mathbb{P}_\mu\{\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3\cap\mathcal{E}_4\} = 1 \tag{32}$$

for any initial distribution $\mu(\cdot) : \mathbb{S} \to [0,1]$ of the Markov chain. At this point, we propose the following decomposition:

$$\begin{aligned}\sum_{j=1}^{n} f(X_j) &= \sum_{j=1}^{T_x-1} f(X_j) + \sum_{m=1}^{K_n-2}\left[\sum_{j=T_x^m}^{T_x^{m+1}-1} f(X_j)\right] + \sum_{j=T_x^{K_n-1}}^{n} f(X_j)\\ &= \sum_{j=1}^{T_x-1} f(X_j) + \sum_{m=1}^{K_n-2} V_m^f + \sum_{j=T_x^{K_n-1}}^{n} f(X_j),\end{aligned}$$

which leads to

$$\frac{1}{K_n}\sum_{j=1}^{n} f(X_j) = \underbrace{\frac{1}{K_n}\sum_{j=1}^{T_x-1} f(X_j)}_{\text{(T1)}} + \underbrace{\left(1-\frac{2}{K_n}\right)\frac{1}{K_n-2}\sum_{m=1}^{K_n-2} V_m^f}_{\text{(T2)}} + \underbrace{\frac{1}{K_n}\sum_{j=T_x^{K_n-1}}^{n} f(X_j)}_{\text{(T3)}}. \tag{33}$$

We remark that if $w \in \mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3\cap\mathcal{E}_4$, $\lim_{n\to\infty} K_n(\omega) = +\infty$. Therefore, it's clear that $\lim_{n\to\infty}\text{(T1)}(\omega) = 0$, because $\omega \in \mathcal{E}_4$ implies $T_x(\omega) < +\infty$, and

$$\lim_{n\to\infty}\text{(T2)}(\omega) = \lim_{n\to\infty}\left(1-\frac{2}{K_n(\omega)}\right)\frac{1}{K_n(\omega)-2}\sum_{m=1}^{K_n(\omega)-2} V_m^f(\omega) = \mathbb{E}_\pi\left[V_1^f\right], \tag{34}$$

since $\omega \in \mathcal{E}_2$. On the other hand,

$$\begin{aligned}|\text{(T3)}(\omega)| &\le \frac{1}{K_n(\omega)}\sum_{j=T_x^{K_n(\omega)-1}(\omega)}^{n} |f(X_j(\omega))|\\ &\overset{\text{(a')}}{\le} \frac{1}{K_n(\omega)}\sum_{j=T_x^{K_n(\omega)-1}(\omega)}^{T_x^{K_n(\omega)}(\omega)} |f(X_j(\omega))|\\ &= \frac{1}{K_n(\omega)} V_{K_n(\omega)-1}^{|f|}(\omega)\\ &\le \frac{1}{K_n(\omega)}\max\left\{V_m^{|f|} : m \in [K_n(\omega)]\right\}\\ &\overset{\text{(b')}}{\to} 0,\end{aligned} \tag{35}$$

where the step (a') holds since $n \leq T_x^{K_n(\omega)}(\omega)$, and the step (b') is owing to the fact $\omega \in \mathcal{E}_3$ together with $\lim_{n\to\infty} K_n(\omega) = +\infty$. Combining the above pieces (34) and (35) together with the fact $\lim_{n\to\infty} (\mathrm{T1})(\omega) = 0$ deduces

$$\lim_{n\to\infty} \frac{1}{K_n(\omega)} \sum_{j=1}^{n} f\left(X_j(\omega)\right) = 0 + \mathbb{E}_\pi\left[V_1^f\right] + 0 = \mathbb{E}_\pi\left[V_1^f\right]$$

for all $w \in \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \cap \mathcal{E}_4$. Hence, for any $w \in \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \cap \mathcal{E}_4$,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f\left(X_j(\omega)\right) &= \lim_{n\to\infty} \frac{K_n(\omega)}{n} \cdot \frac{1}{K_n(\omega)} \sum_{j=1}^{n} f\left(X_j(\omega)\right) \\ &\overset{\text{(c')}}{=} \frac{\mathbb{E}_\pi\left[V_1^f\right]}{\mathbb{E}_x\left[T_x\right]} \\ &= \pi(f), \end{aligned} \tag{36}$$

where the step (c') follows from the fact $\omega \in \mathcal{E}_1$. So, (32) finishes the proof of the statement (iii).

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.