MAS651 Theory of Stochastic Processes
Homework #4

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$. Also, we use the symbol $\mathbb{S}$ instead of S to denote the underlying state space of stochastic processes.

We assume throughout this homework that the underlying state space $\mathbb{S}$ is countable and it is equipped with the discrete σ-field $2^{\mathbb{S}}$ on $\mathbb{S}$. Since $\left(\mathbb{S}, 2^{\mathbb{S}}\right)$ is a *nice measurable space*, it admits the canonical construction in *Section 5.2* in [1] of the probability measure $\mathbb{P}_\mu$ on the sequence space $(\Omega_0, \mathcal{F}_\infty)$ so that the sequence of coordinate maps $\{X_n(\omega) := \omega_n\}_{n=0}^\infty$ is a homogeneous Markov chain with initial distribution μ and transition probability $p(\cdot,\cdot) : \mathbb{S} \times 2^{\mathbb{S}} \to [0,1]$. We remark that it is conventional to write $p(x,y) := p(x,\{y\})$ for $x, y \in \mathbb{S}$.

Problem 1 (*Exercise 5.5.2.* in [1])**.**

Recall that for any given recurrent state $x \in \mathbb{S}$ and any state $y \in \mathbb{S} \setminus \{x\}$,

$$\begin{aligned}
\mu_x(y) &:= \mathbb{E}_x\left[\sum_{n=0}^{T_x-1} \mathbb{1}_{\{X_n=y\}}\right] \\
&\overset{\text{(a)}}{=} \mathbb{E}_x\left[\left(\sum_{n=0}^{T_x-1} \mathbb{1}_{\{X_n=y\}}\right)\mathbb{1}_{\{T_x<+\infty\}}\right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_x\left[\left(\sum_{k=0}^{\infty} \mathbb{1}_{\left\{\sum_{n=0}^{T_x-1}\mathbb{1}_{\{X_n=y\}}=k\right\}}\right)\mathbb{1}_{\{T_x<+\infty\}}\right] \\
&\overset{\text{(c)}}{=} \sum_{k=1}^{\infty} k \cdot \mathbb{P}_x\left\{\sum_{n=0}^{T_x-1} \mathbb{1}_{\{X_n=y\}} = k \text{ and } T_x < +\infty\right\} \\
&\overset{\text{(d)}}{=} \sum_{k=1}^{\infty} k \cdot \mathbb{P}_x\left\{\sum_{n=0}^{T_x-1} \mathbb{1}_{\{X_n=y\}} = k\right\},
\end{aligned} \tag{1}$$

where the above steps (a)–(c) holds since

(a) since the state x is recurrent, $\mathbb{P}_x\{T_x < +\infty\} = 1$;

(b) if $T_x < +\infty$, then $\sum_{n=0}^{T_x-1} \mathbb{1}_{\{X_n=y\}} < +\infty$;

(c) the Fubini-Tonelli's theorem, since the summands are non-negative;

(d) the same reason as the step (a).

Here, $T_x := \inf\{n \geq 1 : X_n = x\}$ denotes the first hitting time to state $x \in \mathbb{S}$.

Claim 1. *For every $k \in \mathbb{N}$, we have*

$$\mathbb{P}_x \left\{ \sum_{n=0}^{T_x - 1} \mathbb{1}_{\{X_n = y\}} = k \right\} = w_{xy} \left(1 - w_{yx}\right)^{k-1} w_{yx}. \tag{2}$$

Proof of Claim 1.

Let $Y_k := \mathbb{1}_{\left\{\sum_{n=0}^{T_x-1} \mathbb{1}_{\{X_n=y\}}=k\right\}}$ for $k \in \mathbb{N}$. It's clear that each Y_k is a bounded measurable function defined on the sequence space $(\Omega_0, \mathcal{F}_\infty)$. If $T_y(\omega) < T_x(\omega)$, then

$$\begin{aligned}
\left(Y_k \circ \theta_{T_y}\right)(\omega) &= \mathbb{1}_{\left\{\sum_{n=0}^{T_x-1} \mathbb{1}_{\{X_n=y\}}=k\right\}} \left(\theta_{T_y}(\omega)\right) \\
&= \begin{cases} 1 & \text{if } \sum_{n=0}^{T_x\left(\theta_{T_y}(\omega)\right)-1} \mathbb{1}_{\left\{X_n\left(\theta_{T_y}(\omega)\right)=1\right\}} = k; \\ 0 & \text{otherwise} \end{cases} \\
&\overset{\text{(e)}}{=} \begin{cases} 1 & \text{if } \sum_{n=0}^{T_x(\omega)-T_y(\omega)-1} \mathbb{1}_{\left\{\omega_{n+T_y(\omega)}=y\right\}} = k; \\ 0 & \text{otherwise} \end{cases} \\
&\overset{\text{(f)}}{=} \begin{cases} 1 & \text{if } \sum_{n=1}^{T_x(\omega)-1} \mathbb{1}_{\{\omega_n=y\}} = k; \\ 0 & \text{otherwise} \end{cases} \\
&= \mathbb{1}_{\left\{\sum_{n=1}^{T_x(\omega)-1} \mathbb{1}_{\{X_n=y\}}=k\right\}}
\end{aligned} \tag{3}$$

where the step (e) holds since when $T_y(\omega) < T_x(\omega)$,

$$\begin{aligned}
T_x\left(\theta_{T_y}(\omega)\right) &= \inf\left\{n \geq 1 : X_n\left(\theta_{T_y}(\omega)\right) = x\right\} \\
&= \inf\left\{n \geq 1 : \omega_{n+T_y(\omega)} = x\right\} \\
&= \inf\left\{n \geq T_y(\omega) + 1 : \omega_n = x\right\} - T_y(\omega) \\
&\overset{\text{(g)}}{=} \inf\left\{n \geq 1 : X_n = x\right\} - T_y(\omega) \\
&= T_x(\omega) - T_y(\omega),
\end{aligned}$$

where the step (g) follows from the fact that $T_y(\omega) < T_x(\omega)$ implies that there is no visits to state x from time 1 to $T_y(\omega)$, and the step (f) is due to the fact that there is no visits to state y from time 1 to $T_y(\omega) - 1$.

Thus,

$$
\begin{aligned}
\mathbb{P}_x\left\{\sum_{n=0}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=k\right\} &\overset{\text{(h)}}{=} \mathbb{P}_x\left\{\sum_{n=1}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=k\right\} \\
&= \mathbb{E}_x\left[\mathbb{1}_{\left\{\sum_{n=1}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=k\right\}}\right] \\
&\overset{\text{(i)}}{=} \mathbb{E}_x\left[\mathbb{1}_{\left\{\sum_{n=1}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=k\right\}}\cdot\mathbb{1}_{\{T_y<T_x\}}\right] \\
&\overset{\text{(j)}}{=} \mathbb{E}_x\left[\left(Y_k\circ\theta_{T_y}\right)\cdot\mathbb{1}_{\{T_y<T_x\}}\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\left(Y_k\circ\theta_{T_y}\right)\cdot\mathbb{1}_{\{T_y<T_x\}}\middle|\mathcal{F}_{T_y}\right]\right] \\
&\overset{\text{(k)}}{=} \mathbb{E}_x\left[\mathbb{E}_x\left[Y_k\circ\theta_{T_y}\middle|\mathcal{F}_{T_y}\right]\mathbb{1}_{\{T_y<T_x\}}\right] \\
&\overset{\text{(l)}}{=} \mathbb{E}_x\left[\mathbb{E}_{X_{T_y}}\left[Y_k\right]\mathbb{1}_{\{T_y<T_x\}}\right] \\
&\overset{\text{(m)}}{=} \mathbb{E}_x\left[\mathbb{E}_y\left[Y_k\right]\mathbb{1}_{\{T_y<T_x\}}\right] \\
&= \mathbb{E}_y\left[Y_k\right]\mathbb{P}_x\left\{T_y<T_x\right\} \\
&= \mathbb{P}_y\left\{\sum_{n=0}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=k\right\}\cdot w_{xy},
\end{aligned}
\tag{4}
$$

where the above steps (h)–(m) can be justified as follows:

(h) $x \neq y$;

(i) $\left\{\sum_{n=1}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=k\right\} \subseteq \{T_y<T_x\}$;

(j) the equality (3);

(k) $\{T_y<T_x\}\in\mathcal{F}_{T_y}$, where $\{\mathcal{F}_n := \sigma\left(X_0, X_1, \cdots, X_n\right)\}_{n=0}^{\infty}$ is the canonical filtration of the Markov chain $\{X_n\}_{n=0}^{\infty}$, since

$$\{T_y<T_x\}\cap\{T_y=n\}=\{T_x>n\}\cap\{T_y=n\}=\left(\Omega_0\setminus\{T_x\le n\}\right)\cap\{T_y=n\}\in\mathcal{F}_n$$

for every $n\in\mathbb{Z}_+$;

(l) the strong Markov property (*Theorem 5.2.5* in [1]);

(m) if $T_y<T_x\le+\infty$, then $X_{T_y}=y$.

Now, we turn our attention to the probability $\mathbb{E}_y\left[Y_k\right]=\mathbb{P}_y\left\{\sum_{n=0}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=k\right\}$. If $k=1$, we find that

$$
\begin{aligned}
\mathbb{E}_y\left[Y_1\right] &= \mathbb{P}_y\left\{\sum_{n=0}^{T_x(\omega)-1}\mathbb{1}_{\{X_n=y\}}=1\right\} \\
&= \mathbb{P}_y\left\{X_n\in\mathbb{S}\setminus\{y\}\text{ for }1\le n\le T_x-1\right\} \\
&= \mathbb{P}_y\left\{T_x<T_y\right\}=w_{yx}.
\end{aligned}
\tag{5}
$$

Hereafter, we assume that $k \geq 2$. Then, we obtain

$$\begin{aligned}
\mathbb{E}_y\left[Y_k\right] &= \mathbb{P}_y\left\{\sum_{n=1}^{T_x(\omega)-1} \mathbb{1}_{\{X_n=y\}} = k-1\right\} \\
&= \mathbb{E}_y\left[\mathbb{1}_{\left\{\sum_{n=1}^{T_x(\omega)-1} \mathbb{1}_{\{X_n=y\}}=k-1\right\}}\right] \\
&\overset{\text{(n)}}{=} \mathbb{E}_y\left[\mathbb{1}_{\left\{\sum_{n=1}^{T_x(\omega)-1} \mathbb{1}_{\{X_n=y\}}=k-1\right\}} \cdot \mathbb{1}_{\{T_y<T_x\}}\right] \\
&\overset{\text{(o)}}{=} \mathbb{E}_y\left[\left(Y_{k-1} \circ \theta_{T_y}\right) \cdot \mathbb{1}_{\{T_y<T_x\}}\right] \\
&= \mathbb{E}_y\left[\mathbb{E}_y\left[\left(Y_{k-1} \circ \theta_{T_y}\right) \cdot \mathbb{1}_{\{T_y<T_x\}} \middle| \mathcal{F}_{T_y}\right]\right] \\
&\overset{\text{(p)}}{=} \mathbb{E}_y\left[\mathbb{E}_y\left[Y_{k-1} \circ \theta_{T_y} \middle| \mathcal{F}_{T_y}\right] \mathbb{1}_{\{T_y<T_x\}}\right] \\
&\overset{\text{(q)}}{=} \mathbb{E}_y\left[\mathbb{E}_{X_{T_y}}\left[Y_{k-1}\right] \mathbb{1}_{\{T_y<T_x\}}\right] \\
&\overset{\text{(r)}}{=} \mathbb{E}_y\left[\mathbb{E}_y\left[Y_{k-1}\right] \mathbb{1}_{\{T_y<T_x\}}\right] \\
&= \mathbb{E}_y\left[Y_{k-1}\right] \underbrace{\mathbb{P}_y\left\{T_y < T_x\right\}}_{=\ 1-\mathbb{P}_y\{T_x<T_y\}} \\
&= \left(1 - w_{yx}\right) \mathbb{E}_y\left[Y_{k-1}\right],
\end{aligned} \tag{6}$$

where the above steps (n)–(r) can be validated as follows:

(n) for $k \geq 2$, $\left\{\sum_{n=1}^{T_x(\omega)-1} \mathbb{1}_{\{X_n=y\}} = k-1\right\} \subseteq \{T_y < T_x\}$;

(o) the equation (3);

(p) the same reason as the step (k);

(q) the strong Markov property (*Theorem 5.2.5* in [1]);

(r) if $T_y < T_x \leq +\infty$, then $X_{T_y} = y$.

So, we may inductively deduce that for every $k \geq 2$,

$$\mathbb{E}_y\left[Y_k\right] = \left(1 - w_{yx}\right)^{k-1} \mathbb{E}_y\left[Y_1\right] \overset{\text{(s)}}{=} \left(1 - w_{yx}\right)^{k-1} w_{yx}, \tag{7}$$

where the step (s) makes use of the equation (5). Putting (7) into the equation (4) yields

$$\mathbb{P}_x\left\{\sum_{n=0}^{T_x(\omega)-1} \mathbb{1}_{\{X_n=y\}} = k\right\} = w_{xy}\mathbb{E}_y\left[Y_k\right] = w_{xy}\left(1 - w_{yx}\right)^{k-1} w_{yx}$$

for all $k \in \mathbb{N}$.

□

Hence, for any state $y \in \mathbb{S} \setminus \{x\}$,

$$\begin{aligned}
\mu_x(y) &= \sum_{k=1}^{\infty} k \cdot \mathbb{P}_x \left\{ \sum_{n=0}^{T_x - 1} \mathbb{1}_{\{X_n = y\}} = k \right\} \\
&\overset{\text{(u)}}{=} \sum_{k=1}^{\infty} k w_{xy} \left(1 - w_{yx}\right)^{k-1} w_{yx} \\
&= w_{xy} w_{yx} \cdot \frac{1}{\left\{1 - (1 - w_{yx})\right\}^2} \\
&= \frac{w_{xy}}{w_{yx}},
\end{aligned}$$

where the step (r) is due to Claim 1. For the case $y = x$, it's clear that $\mu_x(x) = \frac{w_{xx}}{w_{xx}} = 1$ and this completes the proof.

Problem 2 (*Exercise 5.5.3.* in [1]).

Fix any states $x, y \in \mathbb{S}$, and let $\nu_{xy} : \mathbb{S} \to [0, +\infty)$ be defined by

$$\nu_{xy}(z) := \mu_x(y)\mu_y(z), \ \forall z \in \mathbb{S}.$$

Note that $\nu_{xy}(z) < +\infty$ for all $z \in \mathbb{S}$ due to the technical note in the page 303 of [1]. Then for any $w \in \mathbb{S}$, we have

$$\begin{aligned}
\sum_{z \in \mathbb{S}} \nu_{xy}(z) p(z, w) &= \mu_x(y) \sum_{z \in \mathbb{S}} \mu_y(z) p(z, w) \\
&\overset{\text{(a)}}{=} \mu_x(y) \mu_y(w) \\
&= \nu_{xy}(w),
\end{aligned} \tag{8}$$

where the step (a) holds since the state $y \in \mathbb{S}$ is recurrent, $\mu_y(\cdot) : \mathbb{S} \to [0, +\infty)$ is a stationary measure for the transition probability $p(\cdot, \cdot) : \mathbb{S} \times \mathbb{S} \to [0, 1]$ by *Theorem 5.5.7* in [1]. Therefore, $\nu_{xy}(\cdot) : \mathbb{S} \to [0, +\infty)$ is a stationary measure for $p(\cdot, \cdot)$. On the other hand, $\mu_x(\cdot) : \mathbb{S} \to [0, +\infty)$ defines a stationary measure for $p(\cdot, \cdot)$ by *Theorem 5.5.7* in [1]. So, *Theorem 5.5.9* in [1] implies

$$\nu_{xy}(\cdot) = c_{xy} \cdot \mu_x(\cdot) \tag{9}$$

for some constant $c_{xy} \geq 0$. In particular, we obtain

$$\mu_x(y) = \mu_x(y)\mu_y(y) = \nu_{xy}(y) = c_{xy} \cdot \mu_x(y). \tag{10}$$

Lemma 1. *If $p(\cdot, \cdot) : \mathbb{S} \times \mathbb{S} \to [0, 1]$ is irreducible and $\mu(\cdot) : \mathbb{S} \to [0, +\infty]$ is a stationary measure for the transition probability $p(\cdot, \cdot)$ such that $\mu(a) > 0$ for some $a \in \mathbb{S}$, then $\mu(x) > 0$ for all $x \in \mathbb{S}$.*

Proof of Lemma 1.

Assume on the contrary that $\mu(x) = 0$ for some $x \in \mathbb{S} \setminus \{a\}$. Due to the irreducibilty of $p(\cdot, \cdot)$, there is an $N(a, x) \in \mathbb{N}$ such that

$$p^{N(a,x)}(a, x) = \mathbb{P}_a \left\{ X_{N(a,x)} = x \right\} > 0.$$

Since $\mu(\cdot)$ is a stationary measure for $p(\cdot, \cdot)$, we obtain

$$0 = \mu(x) = \sum_{y \in \mathbb{S}} \mu(y) p^{N(a,x)}(y, x) \geq \mu(a) p^{N(a,x)}(a, x), \tag{11}$$

and the last term of the inequality (11), thereby it yields a contradiction! Therefore, $\mu(x) > 0$ for all $x \in \mathbb{S}$. □

Since $\mu_x(x) = 1 > 0$, Lemma 1 implies that $\mu_x(z) > 0$ for all $z \in \mathbb{S}$. So, we can divide the equation (10) by $\mu_x(y)$ and thus we arrive at $c_{xy} = 1$. Hence,

$$\mu_x(z) = \nu_{xy}(z) = \mu_x(y)\mu_y(z)$$

for all $z \in \mathbb{S}$.

Problem 3 (*Exercise 5.5.4.* in [1]).

Fix any state $y \in \mathbb{S}$. If $x = y$, then it's clear from the definition of *positive recurrence* that $\mathbb{E}_y[T_y] < +\infty$. So, we may assume that $x \in \mathbb{S} \setminus \{y\}$.

Claim 2. $\mathbb{E}_x[T_y] \cdot \mathbb{P}_y\{T_x < T_y\} \le \mathbb{E}_y[T_y]$.

Proof of Claim 2.

To begin with, we note that $T_y \in L^1(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_y)$ due to the positive recurrence of state y. One can see that if $T_x(\omega) < T_y(\omega)$, then

$$\begin{aligned}
(T_y \circ \theta_{T_x})(\omega) &= \inf\{n \ge 1 : X_n(\theta_{T_x}(\omega)) = y\} \\
&= \inf\{n \ge T_x(\omega) + 1 : X_n(\omega) = y\} - T_x(\omega) \\
&\overset{\text{(a)}}{=} \inf\{n \ge 1 : X_n(\omega) = y\} - T_x(\omega) \\
&= T_y(\omega) - T_x(\omega),
\end{aligned}$$

where the step (a) holds since if $T_x(\omega) < T_y(\omega)$, then $\omega_n = X_n(\omega) \in \mathbb{S} \setminus \{y\}$ for all $1 \le n \le T_x(\omega)$. In other words,

$$(T_y - T_x) \cdot \mathbb{1}_{\{T_x < T_y\}} = (T_y \circ \theta_{T_x}) \cdot \mathbb{1}_{\{T_x < T_y\}} \tag{12}$$

on Ω_0. Hence, the following holds:

$$\begin{aligned}
\mathbb{E}_y[T_y] &\ge \mathbb{E}_y\left[(T_y - T_x) \cdot \mathbb{1}_{\{T_x < T_y\}}\right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_y\left[(T_y \circ \theta_{T_x}) \cdot \mathbb{1}_{\{T_x < T_y\}}\right] \\
&= \mathbb{E}_y\left[\mathbb{E}_y\left[(T_y \circ \theta_{T_x}) \cdot \mathbb{1}_{\{T_x < T_y\}} \middle| \mathcal{F}_{T_x}\right]\right] \\
&\overset{\text{(c)}}{=} \mathbb{E}_y\left[\mathbb{E}_y\left[T_y \circ \theta_{T_x} \middle| \mathcal{F}_{T_x}\right] \mathbb{1}_{\{T_x < T_y\}}\right] \\
&\overset{\text{(d)}}{=} \mathbb{E}_y\left[\mathbb{E}_{X_{T_x}}[T_y] \, \mathbb{1}_{\{T_x < T_y\}}\right] \\
&\overset{\text{(e)}}{=} \mathbb{E}_y\left[\mathbb{E}_x[T_y] \, \mathbb{1}_{\{T_x < T_y\}}\right] \\
&= \mathbb{E}_x[T_y] \cdot \mathbb{P}_y\{T_x < T_y\},
\end{aligned}$$

where the above steps (b)–(e) can be justified as follows:

(b) the equality (12);

(c) $\{T_x < T_y\} \in \mathcal{F}_{T_x}$, since

$$\{T_x < T_y\} \cap \{T_x = n\} = \{T_y > n\} \cap \{T_x = n\} = (\Omega_0 \setminus \{T_y \le n\}) \cap \{T_x = n\} \in \mathcal{F}_n$$

for every $n \in \mathbb{Z}_+$, where $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^{\infty}$ denotes the canonical filtration of the Markov chain $\{X_n\}_{n=0}^{\infty}$;

(d) the strong Markov property (*Theorem 5.2.5* in [1]);

(e) if $T_x < T_y \le +\infty$, then $X_{T_x} = x$,

and this finishes the proof of Claim 2. □

Claim 3. $\mathbb{P}_y\{T_x < T_y\} > 0$.

Proof of Claim 3.

Due to the irreducibility of $p(\cdot,\cdot)$,

$$K(y,x) := \inf\{n \in \mathbb{N} : p^n(y,x) = \mathbb{P}_y\{X_n = x\} > 0\} < +\infty.$$

Since $\{T_x = K(y,x)\} \subseteq \left\{X_{K(y,x)} = x\right\}$, it's clear that $\mathbb{P}_y\{T_x = K(y,x)\} \le \mathbb{P}_y\left\{X_{K(y,x)} = x\right\}$. On the other hand,

$$\begin{aligned}
\mathbb{P}_y\left\{X_{K(y,x)} = x\right\} &= \mathbb{P}_y\left\{X_{K(y,x)} = x, T_x \le K(y,x)\right\} \\
&= \sum_{l=1}^{K(y,x)-1} \mathbb{P}_y\left\{X_{K(y,x)} = x, T_x = l\right\} + \mathbb{P}_y\left\{X_{K(y,x)} = x, T_x = K(y,x)\right\} \\
&\overset{\text{(f)}}{\le} \sum_{l=1}^{K(y,x)-1} \mathbb{P}_y\left\{X_l = x, X_{K(y,x)} = x\right\} + \mathbb{P}_y\{T_x = K(y,x)\} \\
&\overset{\text{(g)}}{=} \sum_{l=1}^{K(y,x)-1} \underbrace{p^l(y,x)}_{=0} p^{K(y,x)-l}(x,x) + \mathbb{P}_y\{T_x = K(y,x)\} \\
&\overset{\text{(h)}}{=} \mathbb{P}_y\{T_x = K(y,x)\},
\end{aligned}$$

where the above steps (f)–(h) can be validated as follows:

(f) $\{T_x = l\} \subseteq \{X_l = x\}$ for $1 \le l \le K(y,x)$;

(g) a consequence of Chapman-Kolmogorov's equation (*Theorem 5.2.4* in [1]);

(h) due to the minimality of $K(y,x)$ in the set $\{n \in \mathbb{N} : p^n(y,x) = \mathbb{P}_y\{X_n = x\} > 0\}$.

Thus, we obtain $\mathbb{P}_y\{T_x = K(y,x)\} = \mathbb{P}_y\left\{X_{K(y,x)} = x\right\}$. Also one has for every $1 \le l \le K(y,x) - 1$,

$$\begin{aligned}
\mathbb{P}_y\{T_x = K(y,x), T_y < T_x\} &= \sum_{l=1}^{K(y,x)-1} \mathbb{P}_y\{T_x = K(y,x), T_y = l\} \\
&\le \sum_{l=1}^{K(y,x)-1} \mathbb{P}_y\left\{X_l = y, X_{K(y,x)} = x\right\} \\
&\overset{\text{(i)}}{=} \sum_{l=1}^{K(y,x)-1} p^l(y,y) \underbrace{p^{K(y,x)-l}(y,x)}_{=0} \\
&\overset{\text{(j)}}{=} 0,
\end{aligned} \tag{13}$$

where the step (i) is a consequence of Chapman-Kolmogorov's equation (*Theorem 5.2.4* in [1]), and the step (j) follows from the minimality of $K(y,x)$ in the set $\{n \in \mathbb{N} : p^n(y,x) = \mathbb{P}_y\{X_n = x\} > 0\}$. Hence,

$$\begin{aligned}\mathbb{P}_y\{T_x < T_y\} &\geq \mathbb{P}_y\{T_x < T_y, T_x = K(y,x)\} \\ &= \mathbb{P}_y\{T_x = K(y,x)\} - \underbrace{\mathbb{P}_y\{T_y < T_x, T_x = K(y,x)\}}_{=0} \\ &\overset{\text{(k)}}{=} p^{K(y,x)}(y,x) > 0,\end{aligned}$$

where the step (k) makes use of the fact $\mathbb{P}_y\{T_x = K(y,x)\} = \mathbb{P}_y\left\{X_{K(y,x)} = x\right\}$ together with (13), and this establishes our desired claim.

□

Combining Claim 2 together with Claim 3 gives

$$\mathbb{E}_x[T_y] \leq \frac{\mathbb{E}_y[T_y]}{\mathbb{P}_y\{T_x < T_y\}} \overset{\text{(l)}}{<} +\infty,$$

where the step (l) is owing to the positive recurrence of the state $y \in \mathbb{S}$, and this completes the solution to the problem.

Problem 4 (*Exercise 5.5.5.* in [1])**.**

Assume that $p(\cdot,\cdot) : \mathbb{S} \times \mathbb{S} \to [0,1]$ is irreducible and has a stationary measure $\mu(\cdot) : \mathbb{S} \to [0,+\infty]$ such that $\mu \not\equiv +\infty$ on $\mathbb{S}$ and $\sum_{x\in\mathbb{S}} \mu(x) = +\infty$.

Claim 4. *$\mu(x) < +\infty$ for all $x \in \mathbb{S}$.*

Proof of Claim 4.

Assume on the contrary that $\mu(y) = +\infty$ for some $y \in \mathbb{S}$. Due to the irreducibility of $p(\cdot,\cdot)$, for every $x \in \mathbb{S} \setminus \{y\}$, there is a $K(y,x) \in \mathbb{N}$ such that

$$p^{K(y,x)}(y,x) = \mathbb{P}_y\left\{X_{K(y,x)} = x\right\} > 0.$$

Since $\mu(\cdot) : \mathbb{S} \to [0,+\infty]$ is a stationary measure for $p(\cdot,\cdot)$, we have

$$\mu(x) = \sum_{z\in\mathbb{S}} \mu(z) p^{K(y,x)}(z,x) \geq \mu(y) p^{K(y,x)}(y,x) = +\infty,$$

which implies $\mu \equiv +\infty$ on $\mathbb{S}$. This contradicts the assumption that $\mu \not\equiv +\infty$ on $\mathbb{S}$, and finishes the proof.

□

Now, let's prove that $p(\cdot,\cdot)$ is not positive recurrent. Suppose not. Then in particular, $p(\cdot,\cdot)$ is recurrent and so *Theorem 5.5.9* in [1] implies that for each $a \in \mathbb{S}$,

$$\mu(\cdot) = c_a \mu_a(\cdot) \tag{14}$$

on $\mathbb{S}$, for some constant $c_a \in (0, +\infty)$. Thus, one can see that

$$\begin{aligned}
+\infty &= \sum_{x\in\mathbb{S}} \mu(x) = c_a \sum_{x\in\mathbb{S}} \mu_a(x) \\
&= c_a \sum_{x\in\mathbb{S}} \mathbb{E}_a \left[\sum_{n=0}^{T_a-1} \mathbb{1}_{\{X_n = x\}} \right] \\
&= c_a \sum_{x\in\mathbb{S}} \mathbb{E}_a \left[\sum_{n=1}^{T_a} \mathbb{1}_{\{X_n = x\}} \right] \\
&= c_a \sum_{x\in\mathbb{S}} \mathbb{E}_a \left[\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = x\}} \cdot \mathbb{1}_{\{n \le T_a\}} \right] \\
&\overset{\text{(a)}}{=} c_a \sum_{x\in\mathbb{S}} \left[\sum_{n=1}^{\infty} \mathbb{P}_a \{X_n = x, n \le T_a\} \right] \\
&\overset{\text{(b)}}{=} c_a \sum_{n=1}^{\infty} \left[\sum_{x\in\mathbb{S}} \mathbb{P}_a \{X_n = x, n \le T_a\} \right] \\
&= c_a \sum_{n=1}^{\infty} \mathbb{P}_a \{T_a \ge n\} \\
&= c_a \sum_{n=1}^{\infty} \mathbb{E}_a \left[\mathbb{1}_{\{n \le T_a\}} \right] \\
&\overset{\text{(c)}}{=} c_a \mathbb{E}_a \left[\sum_{n=1}^{\infty} \mathbb{1}_{\{n \le T_a\}} \right] \\
&= c_a \mathbb{E}_a [T_a] \overset{\text{(d)}}{<} +\infty,
\end{aligned}$$

which gives a contradiction. Here, the above steps (a)–(d) holds since:

(a) we can change the order of expectation and summation by the monotone convergence theorem;

(b) the Fubini-Tonelli's theorem;

(c) the same reason as the step (a);

(d) we have assumed that $p(\cdot, \cdot)$ is positive recurrent.

Hence, we can conclude that $p(\cdot, \cdot)$ is not positive recurrent.

Problem 5 (*Exercise 5.5.9.* in [1]).

Suppose that $\{X_n\}_{n=0}^{\infty}$ is an $\mathbb{S}$-valued homogeneous Markov chain with transition probability $p(\cdot, \cdot) : \mathbb{S} \times \mathbb{S} \to [0, 1]$ such that $\mathbb{E}_x [X_1] \le x - \epsilon$ for all $x > K$, where $\mathbb{S} \subseteq [0, +\infty)$ is a countable state space and $\epsilon > 0$. We may assume that the given Markov chain is the sequence of coordinate maps $\{X_n(\omega) := \omega_n\}_{n=0}^{\infty}$ defined on the sequence space $(\Omega_0, \mathcal{F}_\infty)$, where the canonical construction of the probability measure $\mathbb{P}_\mu$ on $(\Omega_0, \mathcal{F}_\infty)$ in *Section 5.2* of [1] makes it an $\mathbb{S}$-valued homogeneous Markov chain with initial distribution μ and transition probability $p(\cdot, \cdot)$. Now, we will prove the following result:

Claim 5. *$\{Y_{n\wedge\tau}\}_{n=0}^{\infty}$ is a positive supermartingale with respect to the canonical filtration $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^{\infty}$ on the canonical probability space $(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_\mu)$, where $\mu(\cdot)$ is a probability measure on $\left(\mathbb{S}, 2^{\mathbb{S}}\right)$ with finite first moment, i.e., $\sum_{x\in\mathbb{S}} x\mu(x) < +\infty$.*

Proof of Claim 5.

We first prove the $\mathbb{P}_\mu$-integrability of $Y_{n\wedge\tau}$. From the inequality

$$\begin{aligned} Y_{n\wedge\tau} &= Y_n \cdot \mathbb{1}_{\{\tau\geq n\}} + \sum_{k=0}^{n-1} Y_k \cdot \mathbb{1}_{\{\tau=k\}} \\ &= Y_n \cdot \mathbb{1}_{\{\tau\geq n\}} + \sum_{k=0}^{n-1} (X_k + k\epsilon)\, \mathbb{1}_{\{\tau=k\}} \\ &\overset{\text{(a)}}{\leq} Y_n \cdot \mathbb{1}_{\{\tau\geq n\}} + \sum_{k=0}^{n-1} (K + k\epsilon)\, \mathbb{1}_{\{\tau=k\}} \\ &\leq Y_n \cdot \mathbb{1}_{\{\tau\geq n\}} + (K + n\epsilon)\,, \end{aligned}$$

where the step (a) holds since if $\tau = k$, then $X_k \leq K$, it suffices to show that $Y_n \cdot \mathbb{1}_{\{\tau\geq n\}} \in L^1(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_\mu)$. Choose any $A_0, A_1, \cdots, A_{n-1} \subseteq \mathbb{S}$. Then, we have

$$\begin{aligned} &\mathbb{E}_\mu\left[\left(Y_n \cdot \mathbb{1}_{\{\tau\geq n\}}\right) \mathbb{1}_{\{X_0\in A_0, X_1\in A_1,\cdots,X_{n-1}\in A_{n-1}\}}\right] \\ &\overset{\text{(b)}}{=} \mathbb{E}_\mu\left[(X_n + n\epsilon) \prod_{j=0}^{n-1} \mathbb{1}_{A_j\cap(K,+\infty)}(X_j)\right] \\ &\overset{\text{(c)}}{=} \sum_{x_0\in A_0\cap(K,+\infty)} \mu(x_0) \left[\sum_{x_1\in A_1\cap(K,+\infty)} p(x_0, x_1)\right. \\ &\qquad \left.\left[\cdots\left[\sum_{x_{n-1}\in A_{n-1}\cap(K,+\infty)} p(x_{n-2}, x_{n-1}) \underbrace{\left[\sum_{x_n\in\mathbb{S}} p(x_{n-1}, x_n)\,(x_n + n\epsilon)\right]}_{=\,\mathbb{E}_{x_{n-1}}[X_1+n\epsilon]}\right]\cdots\right]\right] \qquad (15) \\ &\overset{\text{(d)}}{\leq} \sum_{x_0\in A_0\cap(K,+\infty)} \mu(x_0) \left[\sum_{x_1\in A_1\cap(K,+\infty)} p(x_0, x_1)\right. \\ &\qquad \left.\left[\cdots\left[\sum_{x_{n-1}\in A_{n-1}\cap(K,+\infty)} p(x_{n-2}, x_{n-1})\, \{x_{n-1} + (n-1)\epsilon\}\right]\cdots\right]\right] \\ &\overset{\text{(e)}}{=} \mathbb{E}_\mu\left[\{X_{n-1} + (n-1)\epsilon\} \prod_{j=0}^{n-1} \mathbb{1}_{A_j\cap(K,+\infty)}(X_j)\right] \\ &= \mathbb{E}_\mu\left[\left(Y_{n-1} \cdot \mathbb{1}_{\{\tau\geq n\}}\right) \mathbb{1}_{\{X_0\in A_0, X_1\in A_1,\cdots,X_{n-1}\in A_{n-1}\}}\right], \end{aligned}$$

where the above steps (b)–(e) can be validated as follows:

(b) $\{\tau \geq n\} = \{X_0 > K, X_1 > K, \cdots, X_{n-1} > K\}$;

(c) the *equation (5.2.3)* in [1];

(d) since $x_{n-1} > K$, we have $\mathbb{E}_{x_{n-1}}[X_1] \leq x_{n-1} - \epsilon$;

(e) the same reason as the step (c).

Putting $A_0 = A_1 = \cdots = A_{n-1} = \mathbb{S}$ into the bound (15) yields

$$\mathbb{E}_\mu \left[Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \right] \le \mathbb{E}_\mu \left[Y_{n-1} \cdot \mathbb{1}_{\{\tau \ge n\}} \right] \le \mathbb{E}_\mu \left[Y_{n-1} \cdot \mathbb{1}_{\{\tau \ge n-1\}} \right] \tag{16}$$

for every $n \in \mathbb{N}$. Applying the inequality (16) inductively, we may deduce that

$$\mathbb{E}_\mu \left[Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \right] \le \mathbb{E}_\mu \left[Y_0 \cdot \mathbb{1}_{\{\tau \ge 0\}} \right] = \mathbb{E}_\mu \left[X_0 \right] = \sum_{x \in \mathbb{S}} x\mu(x) \overset{\text{(f)}}{<} +\infty,$$

where the step (f) is due to the assumption of the initial distribution $\mu(\cdot)$, thereby $Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \in L^1(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_\mu)$. This establishes the $\mathbb{P}_\mu$-integrability of $Y_{n \wedge \tau}$.

Now, put $A_0 = \{x_0\}, A_1 = \{x_1\}, \cdots, A_{n-1} = \{x_{n-1}\}$ for $x_0, x_1, \cdots, x_{n-1} \in \mathbb{S}$. Then, we obtain

$$\mathbb{E}_\mu \left[\left(Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \right) \mathbb{1}_{\{(X_0, X_1, \cdots, X_{n-1}) = (x_0, x_1, \cdots, x_{n-1})\}} \right] \le \mathbb{E}_\mu \left[\left(Y_{n-1} \cdot \mathbb{1}_{\{\tau \ge n\}} \right) \mathbb{1}_{\{(X_0, X_1, \cdots, X_{n-1}) = (x_0, x_1, \cdots, x_{n-1})\}} \right] \tag{17}$$

for any $(x_0, x_1, \cdots, x_{n-1}) \in \mathbb{S}^n$. So given any $B \in \mathcal{S}^n$, one has

$$\begin{aligned} &\mathbb{E}_\mu \left[\left(Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \right) \mathbb{1}_{\{(X_0, X_1, \cdots, X_{n-1}) \in B\}} \right] \\ &= \sum_{(x_0, x_1, \cdots, x_{n-1}) \in B} \mathbb{E}_\mu \left[\left(Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \right) \mathbb{1}_{\{(X_0, X_1, \cdots, X_{n-1}) = (x_0, x_1, \cdots, x_{n-1})\}} \right] \\ &\overset{\text{(g)}}{\le} \sum_{(x_0, x_1, \cdots, x_{n-1}) \in B} \mathbb{E}_\mu \left[\left(Y_{n-1} \cdot \mathbb{1}_{\{\tau \ge n\}} \right) \mathbb{1}_{\{(X_0, X_1, \cdots, X_{n-1}) = (x_0, x_1, \cdots, x_{n-1})\}} \right] \\ &= \mathbb{E}_\mu \left[\left(Y_{n-1} \cdot \mathbb{1}_{\{\tau \ge n\}} \right) \mathbb{1}_{\{(X_0, X_1, \cdots, X_{n-1}) \in B\}} \right], \end{aligned}$$

where the step (g) follows from the bound (17). From the fact $\{\tau \ge n\} = \Omega_0 \in \{\tau \le n-1\} \in \mathcal{F}_{n-1}$, which holds since τ is a stopping time with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^\infty$, one can conclude that

$$\mathbb{E}_\mu \left[Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \middle| \mathcal{F}_{n-1} \right] \overset{\mathbb{P}_\mu\text{-a.s.}}{\le} Y_{n-1} \cdot \mathbb{1}_{\{\tau \ge n\}} \tag{18}$$

for every $n \in \mathbb{N}$. Therefore, we finally arrive at

$$\begin{aligned} \mathbb{E}_\mu \left[Y_{n \wedge \tau} \middle| \mathcal{F}_{n-1} \right] &\overset{\text{(h)}}{=} \mathbb{E}_\mu \left[Y_n \cdot \mathbb{1}_{\{\tau \ge n\}} \middle| \mathcal{F}_{n-1} \right] + \sum_{k=0}^{n-1} Y_k \cdot \mathbb{1}_{\{\tau = k\}} \\ &\overset{\text{(i)}}{\le} Y_{n-1} \cdot \mathbb{1}_{\{\tau \ge n\}} + \sum_{k=0}^{n-1} Y_k \cdot \mathbb{1}_{\{\tau = k\}} \\ &= Y_{(n-1) \wedge \tau} \end{aligned}$$

$\mathbb{P}_\mu$-almost surely, where the step (h) holds since for every $k \in [0 : n-1]$, $Y_k \cdot \mathbb{1}_{\{\tau = k\}} \in \mathcal{F}_k \subseteq \mathcal{F}_{n-1}$, and the step (i) makes use of the inequality (18). Hence, $\{Y_{n \wedge \tau}\}_{n=0}^\infty$ is a positive supermartingale with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ on the canonical probability space $(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_\mu)$, where $\mu(\cdot)$ is a probability measure on $\left(\mathbb{S}, 2^{\mathbb{S}}\right)$ with $\sum_{x \in \mathbb{S}} x\mu(x) < +\infty$.

□

Note that every Dirac measure centered on some fixed state $x \in \mathbb{S}$, $\delta_x(\cdot)$, satisfies the condition of Claim 5. Indeed,

$$\sum_{y \in \mathbb{S}} y \cdot \delta_x(y) = x < +\infty$$

for all $x \in \mathbb{S}$. Thus, $\{Y_{n\wedge\tau}\}_{n=0}^{\infty}$ is a positive supermartingale with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$ on the canonical probability space $(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_x)$, for every state $x \in \mathbb{S}$. Due to the optional stopping theorem for non-negative supermartingales (*Theorem 4.8.4* in [1]), one has

$$\begin{aligned} x &= \mathbb{E}_x\left[X_0\right] = \mathbb{E}_x\left[Y_{0\wedge\tau}\right] \\ &\geq \mathbb{E}_x\left[Y_{\tau\wedge\tau}\right] \\ &= \mathbb{E}_x\left[Y_\tau\right] \\ &= \mathbb{E}_x\left[X_\tau + \tau\epsilon\right] \\ &\geq \epsilon \cdot \mathbb{E}_x\left[\tau\right], \end{aligned}$$

thereby $\mathbb{E}_x\left[\tau\right] \leq \frac{x}{\epsilon}$ for all $x \in \mathbb{S}$. This completes the solution to Problem 5.

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.