MAS651 Theory of Stochastic Processes
Homework #3

20150597 Jeonghwan Lee

Department of Mathematical Sciences, KAIST

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$. Also, we use the symbol $\mathbb{S}$ instead of S to denote the underlying state space of stochastic processes.

We assume throughout this homework that the underlying state space $\mathbb{S}$ is countable and it is equipped with the discrete σ-field $2^{\mathbb{S}}$ on $\mathbb{S}$. Since $\left(\mathbb{S}, 2^{\mathbb{S}}\right)$ is a *nice measurable space*, it admits the canonical construction in *Section 5.2* in [1] of the probability measure $\mathbb{P}_\mu$ on the sequence space $(\Omega_0, \mathcal{F}_\infty)$ so that the sequence of coordinate maps $\{X_n(\omega) := \omega_n\}_{n=0}^{\infty}$ is a homogeneous Markov chain with initial distribution μ and transition probability $p(\cdot,\cdot) : \mathbb{S} \times 2^{\mathbb{S}} \to [0,1]$. We remark that it is conventional to write $p(x,y) := p(x, \{y\})$ for $x, y \in \mathbb{S}$.

Problem 1 (*Exercise 5.3.1.* in [1])**.**

Let us use the symbol V_k instead of v_k for each $k \in \mathbb{N}$ to denote the random vectors of our interest. Let

$$\mathbb{V} := \bigcup_{n=1}^{\infty} \left(\{n\} \times \mathbb{S}^n\right)$$

denote the state space of random vectors $\{V_k : k \in \mathbb{N}\}$, and note that $\mathbb{V}$ is a countable set. In order to show that $\{V_k : k \in \mathbb{N}\}$ are independent under the canonical probability space $(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_y)$, where $y \in \mathbb{S}$ is a recurrent space of the given Markov chain, if suffices to show that

$$\mathbb{P}_y\left\{V_{k_1} = v_{k_1}, V_{k_2} = v_{k_2}, \cdots, V_{k_r} = v_{k_r}\right\} = \prod_{j=1}^{r} \mathbb{P}_y\left\{V_{k_j} = v_{k_j}\right\} \tag{1}$$

for all $r \in \mathbb{N}$, $1 \leq k_1 < k_2 < \cdots < k_r < +\infty$, and $v_{k_1}, v_{k_2}, \cdots, v_{k_r} \in \mathbb{V}$.

Now fix any $k \geq 2$ and $v := (n, x_0, x_1, \cdots, x_{n-1}) \in \mathbb{V}$. Define $Y : \Omega_0 \to \mathbb{R}$ by

$$Y(\omega) := \mathbb{1}_{\{V_1 = v\}}(\omega) = \begin{cases} 1 & \text{if } r_1(\omega) = n, X_0(\omega) = x_0, X_1(\omega) = x_1, \cdots, X_{r_1(\omega)-1}(\omega) = x_{n-1}; \\ 0 & \text{otherwise.} \end{cases}$$

It's clear that Y is a bounded measurable function from $(\Omega_0, \mathcal{F}_\infty)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel

σ-field on $\mathbb{R}$. Also if $R_{k-1}(\omega) < +\infty$, we find that

$$\left(Y \circ \theta_{R_{k-1}}\right)(\omega) = \begin{cases} 1 & \text{if } r_1\left(\theta_{R_{k-1}}(\omega)\right) = n, X_0\left(\theta_{R_{k-1}}(\omega)\right) = x_0, \cdots, X_{r_1(\omega)-1}\left(\theta_{R_{k-1}}(\omega)\right) = x_{n-1}; \\ 0 & \text{otherwise.} \end{cases}$$
$$\stackrel{\text{(a)}}{=} \begin{cases} 1 & \text{if } r_k(\omega) = n, X_{R_{k-1}(\omega)}(\omega) = x_0, \cdots, X_{R_{k-1}(\omega)+r_k(\omega)-1}(\omega) = x_{n-1}; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$
$$= \mathbb{1}_{\{V_k = v\}}(\omega),$$

where the step (a) holds since when $R_{k-1}(\omega) < +\infty$,

$$\begin{aligned} \left(r_1 \circ \theta_{R_{k-1}}\right)(\omega) &= \inf\left\{n > 0 : X_n\left(\theta_{R_{k-1}}(\omega)\right) = y\right\} \\ &= \inf\left\{n > 0 : X_{n+R_{k-1}(\omega)}(\omega) = y\right\} \\ &= R_k(\omega) - R_{k-1}(\omega) \\ &= r_k(\omega). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbb{P}_y\left\{V_k = v \middle| \mathcal{F}_{R_{k-1}}\right\} &= \mathbb{E}_y\left[\mathbb{1}_{\{V_k=v\}} \middle| \mathcal{F}_{R_{k-1}}\right] \\ &\stackrel{\text{(b)}}{=} \mathbb{E}_y\left[\mathbb{1}_{\{V_k=v\}} \middle| \mathcal{F}_{R_{k-1}}\right] \cdot \mathbb{1}_{\{R_{k-1}<+\infty\}} \\ &\stackrel{\text{(c)}}{=} \mathbb{E}_y\left[\mathbb{1}_{\{V_k=v\}} \cdot \mathbb{1}_{\{R_{k-1}<+\infty\}} \middle| \mathcal{F}_{R_{k-1}}\right] \\ &\stackrel{\text{(d)}}{=} \mathbb{E}_y\left[\left(Y \circ \theta_{R_{k-1}}\right) \cdot \mathbb{1}_{\{R_{k-1}<+\infty\}} \middle| \mathcal{F}_{R_{k-1}}\right] \\ &\stackrel{\text{(e)}}{=} \mathbb{E}_y\left[Y \circ \theta_{R_{k-1}} \middle| \mathcal{F}_{R_{k-1}}\right] \cdot \mathbb{1}_{\{R_{k-1}<+\infty\}} \\ &\stackrel{\text{(f)}}{=} \mathbb{E}_{X_{R_{k-1}}}\left[Y\right] \cdot \mathbb{1}_{\{R_{k-1}<+\infty\}} \\ &\stackrel{\text{(g)}}{=} \mathbb{E}_y\left[Y\right] = \mathbb{P}_y\left\{V_1 = v\right\} \end{aligned} \tag{3}$$

$\mathbb{P}_y$-almost surely, where the above steps (b)–(g) can be validated as follows:

(b) since $y \in \mathbb{S}$ is a recurrent state, $\mathbb{P}_y\{R_n < +\infty\} = 1$ for all $n \in \mathbb{Z}_+$ by *Theorem 5.2.6* in [1];

(c) $\{R_{k-1} < +\infty\} \in \mathcal{F}_{R_{k-1}}$, because

$$\{R_{k-1} < +\infty\} \cap \{R_{k-1} = n\} = \{R_{k-1} = n\} \in \mathcal{F}_n$$

for all $n \in \mathbb{Z}_+$;

(d) the equation (2);

(e) the same reason as the step (c);

(f) the strong Markov property (*Theorem 5.2.5* in [1]);

(g) if $R_{k-1} < +\infty$, then $X_{R_{k-1}} = y$ and since $y \in \mathbb{S}$ is a recurrent state, $\mathbb{P}_y\{R_n < +\infty\} = 1$ for all $n \in \mathbb{Z}_+$ by *Theorem 5.2.6* in [1].

Here, $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^{\infty}$ is the canonical filtration of the Markov chain $\{X_n\}_{n=0}^{\infty}$. In particular, we can deduce

$$\mathbb{P}_y\{V_k = v\} = \mathbb{E}_y\left[\mathbb{P}_y\left\{V_k = v \mid \mathcal{F}_{R_{k-1}}\right\}\right] = \mathbb{P}_y\{V_1 = v\} \tag{4}$$

for all $k \in \mathbb{N}$ and $v \in \mathbb{V}$, thereby $\{V_k : k \in \mathbb{N}\}$ are identically distributed.

Finally, we prove the independence of the sequence $\{V_k\}_{k=1}^{\infty}$. Choose any sequence $\{v_k\}_{k=1}^{\infty}$ in $\mathbb{V}$, where

$$v_k = \left(n_k, x_0^{(k)}, x_1^{(k)}, \cdots, x_{n_k-1}^{(k)}\right), \ \forall k \in \mathbb{N}.$$

Then, we can make the following observations:

$$\{V_1 = v_1, \cdots, V_{k-1} = v_{k-1}\} \cap \{R_{k-1} = n\} = \varnothing \in \mathcal{F}_n \tag{5}$$

for all $n \in \mathbb{Z}_+ \setminus \left\{\sum_{j=1}^{k-1} n_j\right\}$, and

$$\begin{aligned}
&\{V_1 = v_1, \cdots, V_{k-1} = v_{k-1}\} \cap \left\{R_{k-1} = \sum_{j=1}^{k-1} n_j\right\} \\
&= \bigcap_{j=1}^{k-1} \{V_j = v_j\} \\
&= \bigcap_{j=1}^{k-1} \left\{R_j - R_{j-1} = n_j, X_{R_{j-1}} = x_0^{(j)}, X_{R_{j-1}+1} = x_1^{(j)}, \cdots, X_{R_j - 1} = x_{n_j-1}^{(j)}\right\} \\
&= \bigcap_{j=1}^{k-1} \underbrace{\left(\left\{R_j = \sum_{i=1}^{j} n_i\right\} \cap \left\{X_{\sum_{i=1}^{j-1} n_i} = x_0^{(j)}, X_{\sum_{i=1}^{j-1} n_i + 1} = x_1^{(j)}, \cdots, X_{\sum_{i=1}^{j} n_i - 1} = x_{n_j-1}^{(j)}\right\}\right)}_{\in\, \mathcal{F}_{\sum_{i=1}^{j} n_i} \subseteq \mathcal{F}_{\sum_{i=1}^{k-1} n_i}} \\
&\overset{\text{(h)}}{\in} \mathcal{F}_{\sum_{i=1}^{k-1} n_i},
\end{aligned} \tag{6}$$

where the step (h) holds since

$$\begin{aligned}
\left\{R_j = \sum_{i=1}^{j} n_i\right\} &\in \mathcal{F}_{\sum_{i=1}^{j} n_i} \subseteq \mathcal{F}_{\sum_{i=1}^{k-1} n_i}; \\
\left\{X_{\sum_{i=1}^{j-1} n_i} = x_0^{(j)}, X_{\sum_{i=1}^{j-1} n_i + 1} = x_1^{(j)}, \cdots, X_{\sum_{i=1}^{j} n_i - 1} = x_{n_j-1}^{(j)}\right\} &\in \sigma\left(X_{\sum_{i=1}^{j-1} n_i}, X_{\sum_{i=1}^{j-1} n_i + 1}, \cdots, X_{\sum_{i=1}^{j} n_i - 1}\right) \\
&\subseteq \mathcal{F}_{\sum_{i=1}^{j} n_i} \subseteq \mathcal{F}_{\sum_{i=1}^{k-1} n_i}
\end{aligned}$$

for every $j \in [k-1]$. Combining two observations (5) and (6) concludes $\{V_1 = v_1, \cdots, V_{k-1} = v_{k-1}\} \in \mathcal{F}_{R_{k-1}}$. Hence, we reach

$$\begin{aligned}
\mathbb{P}_y\{V_1 = v_1, \cdots, V_{k-1} = v_{k-1}, V_k = v_k\} &= \int_{\{V_1 = v_1, \cdots, V_{k-1} = v_{k-1}\}} \mathbb{1}_{\{V_k = v_k\}} \mathrm{d}\mathbb{P}_y \\
&\overset{\text{(i)}}{=} \int_{\{V_1 = v_1, \cdots, V_{k-1} = v_{k-1}\}} \mathbb{P}_y\{V_k = v_k\} \mathrm{d}\mathbb{P}_y \\
&= \mathbb{P}_y\{V_1 = v_1, \cdots, V_{k-1} = v_{k-1}\} \mathbb{P}_y\{V_k = v_k\},
\end{aligned} \tag{7}$$

where the step (i) follows from (3) together with (4) for $v = v_k$. Therefore, we can deduce inductively from (7) that

$$\mathbb{P}_y \{V_1 = v_1, V_2 = v_2, \cdots, V_k = v_k\} = \prod_{j=1}^{k} \mathbb{P}_y \{V_j = v_j\} \tag{8}$$

for all $k \in \mathbb{N}$ and $v_1, v_2, \cdots, v_k \in \mathbb{V}$, and this immediately yields our desired result (1). Hence, $\{V_k : k \in \mathbb{N}\}$ is a sequence of independent and identically distributed random vectors.

Problem 2 (*Exercise 5.3.2.* in [1]).

Fix any states $x, y, z \in \mathbb{S}$. Define a function $Z : \Omega_0 \to \mathbb{R}$ by

$$Z(\omega) := \mathbb{1}_{\{T_z < +\infty\}}(\omega) = \mathbb{1}_{\{X_n = z \text{ for some } n > 0\}}(\omega).$$

Then, Z is a bounded measurable function on the sequence space $(\Omega_0, \mathcal{F}_\infty)$ and if $T_y(\omega) < +\infty$, we find that

$$\begin{aligned} \left(Z \circ \theta_{T_y}\right)(\omega) &= \mathbb{1}_{\{X_n = z \text{ for some } n > 0\}} \left(\theta_{T_y}(\omega)\right) \\ &= \mathbb{1}_{\{X_n = z \text{ for some } n > T_y\}}(\omega) \\ &\le \mathbb{1}_{\{X_n = z \text{ for some } n > 0\}}(\omega) = Z(\omega). \end{aligned} \tag{9}$$

Thus, we obtain

$$\left(Z \circ \theta_{T_y}\right) \cdot \mathbb{1}_{\{T_y < +\infty\}} \le Z \cdot \mathbb{1}_{\{T_y < +\infty\}} \tag{10}$$

on Ω_0. Hence, we conclude that

$$\begin{aligned} \rho_{xz} &= \mathbb{P}_x \{T_z < +\infty\} = \mathbb{E}_x [Z] \\ &\ge \mathbb{E}_x \left[Z \cdot \mathbb{1}_{\{T_y < +\infty\}}\right] \\ &\overset{\text{(a)}}{\ge} \mathbb{E}_x \left[\left(Z \circ \theta_{T_y}\right) \cdot \mathbb{1}_{\{T_y < +\infty\}}\right] \\ &= \mathbb{E}_x \left[\mathbb{E}_x \left[\left(Z \circ \theta_{T_y}\right) \cdot \mathbb{1}_{\{T_y < +\infty\}} \middle| \mathcal{F}_{T_y}\right]\right] \\ &\overset{\text{(b)}}{=} \mathbb{E}_x \left[\mathbb{E}_x \left[Z \circ \theta_{T_y} \middle| \mathcal{F}_{T_y}\right] \mathbb{1}_{\{T_y < +\infty\}}\right] \\ &\overset{\text{(c)}}{=} \mathbb{E}_x \left[\mathbb{E}_{X_{T_y}} [Z] \cdot \mathbb{1}_{\{T_y < +\infty\}}\right] \\ &\overset{\text{(d)}}{=} \mathbb{E}_x \left[\mathbb{E}_y [Z] \cdot \mathbb{1}_{\{T_y < +\infty\}}\right] \\ &= \mathbb{P}_x \{T_y < +\infty\} \mathbb{P}_y \{T_z < +\infty\} \\ &= \rho_{xy} \rho_{yz}, \end{aligned}$$

where the steps (a)–(d) can be justified as follows:

(a) the inequality (10);

(b) $\{T_y < +\infty\} \in \mathcal{F}_{T_y}$, since

$$\{T_y < +\infty\} \cap \{T_y = n\} = \{T_y = n\} \in \mathcal{F}_n$$

for all $n \in \mathbb{Z}_+$;

(c) the Strong Markov property (*Theorem 5.2.5* in [1]);

(d) we have $X_{T_y} = y$ when $T_y < +\infty$.

This completes the proof of the desired result.

Problem 3 (*Exercise 5.3.5.* in [1])**.**

We first note that "$\varphi(x) \to 0$ as $x \to +\infty$" means that every *superlevel set* $\mathcal{L}_M^+(\varphi) := \{x \in \mathbb{S} : \varphi(x) \geq M\}$ is finite for all $M \in (0, +\infty)$, while "$\varphi(x) \to +\infty$ as $x \to +\infty$" indicates that every *sublevel set* $\mathcal{L}_M^-(\varphi) := \{x \in \mathbb{S} : \varphi(x) \leq M\}$ is finite for all $M \in (0, +\infty)$. Also, if the state space $\mathbb{S}$ is finite, then the given Markov chain $\{X_n\}_{n=0}^{\infty}$ is recurrent due to *Theorem 5.3.3* in [1]. Therefore, we may assume that the state space $\mathbb{S}$ is countably infinite. Let $\delta := \min\{\varphi(x) : x \in F\} > 0$, and define $\Phi(x) : \mathbb{S} \to [0, +\infty)$ by $\Phi(x) := \frac{\varphi(x)}{\delta}$ for $x \in \mathbb{S}$. One can see that the function Φ satisfies the following properties:

(P1) $\mathbb{E}_x[\Phi(X_1)] = \sum_{y \in \mathbb{S}} p(x, y)\Phi(y) \leq \Phi(x)$ for all $x \in \mathbb{S} \setminus F$;

(P2) $\Phi(x) \geq 1$ for all $x \in F$.

Claim 1. *Let $V_F := \inf\{n \in \mathbb{Z}_+ : X_n \in F\}$ be the first visiting time to $F \subseteq \mathbb{S}$ of the Markov chain $\{X_n\}_{n=0}^{\infty}$. Then, we have*

$$\Phi(x) \geq \mathbb{P}_x\{V_F < +\infty\}, \ \forall x \in \mathbb{S}. \tag{11}$$

Proof of Claim 1.

To begin with, we fix any $x_0 \in \mathbb{S} \setminus F$. We claim that the following bound holds for every $n \in \mathbb{N}$:

$$\begin{aligned} \Phi(x_0) \geq \sum_{k=1}^{n} & \left[\sum_{x_k \in F} \Phi(x_k) \left\{ \sum_{(x_1, x_2, \cdots, x_{k-1}) \in (\mathbb{S} \setminus F)^{k-1}} p(x_0, x_1)p(x_1, x_2) \cdots p(x_{k-1}, x_k) \right\} \right] \\ & + \sum_{x_n \in \mathbb{S} \setminus F} \Phi(x_n) \left\{ \sum_{(x_1, x_2, \cdots, x_{n-1}) \in (\mathbb{S} \setminus F)^{n-1}} p(x_0, x_1)p(x_1, x_2) \cdots p(x_{n-1}, x_n) \right\}. \end{aligned} \tag{12}$$

We proceed the proof of the above claim by induction on n. The case $n = 1$ is immediate from the property

(P1) of the function Φ. We now assume that the above claim holds for $n = l-1$, where $l \geq 2$. Then,

$$\begin{aligned}
\Phi(x_0) \geq\ & \sum_{k=1}^{l-1}\left[\sum_{x_k\in F}\Phi(x_k)\left\{\sum_{(x_1,x_2,\cdots,x_{k-1})\in(\mathbb{S}\setminus F)^{k-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{k-1},x_k)\right\}\right] \\
& + \sum_{x_{l-1}\in\mathbb{S}\setminus F}\Phi(x_{l-1})\left\{\sum_{(x_1,x_2,\cdots,x_{l-2})\in(\mathbb{S}\setminus F)^{l-2}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{l-2},x_{l-1})\right\} \\
\overset{\text{(a)}}{\geq}\ & \sum_{k=1}^{l-1}\left[\sum_{x_k\in F}\Phi(x_k)\left\{\sum_{(x_1,x_2,\cdots,x_{k-1})\in(\mathbb{S}\setminus F)^{k-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{k-1},x_k)\right\}\right] \\
& + \sum_{x_{l-1}\in\mathbb{S}\setminus F}\left\{\sum_{x_l\in\mathbb{S}} p(x_{l-1},x_l)\Phi(x_l)\right\}\left\{\sum_{(x_1,x_2,\cdots,x_{l-2})\in(\mathbb{S}\setminus F)^{l-2}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{l-2},x_{l-1})\right\} \\
=\ & \sum_{k=1}^{l-1}\left[\sum_{x_k\in F}\Phi(x_k)\left\{\sum_{(x_1,x_2,\cdots,x_{k-1})\in(\mathbb{S}\setminus F)^{k-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{k-1},x_k)\right\}\right] \\
& + \sum_{x_{l-1}\in\mathbb{S}\setminus F}\left\{\sum_{x_l\in F} p(x_{l-1},x_l)\Phi(x_l)\right\}\left\{\sum_{(x_1,x_2,\cdots,x_{l-2})\in(\mathbb{S}\setminus F)^{l-2}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{l-2},x_{l-1})\right\} \\
& + \sum_{x_{l-1}\in\mathbb{S}\setminus F}\left\{\sum_{x_l\in\mathbb{S}\setminus F} p(x_{l-1},x_l)\Phi(x_l)\right\}\left\{\sum_{(x_1,x_2,\cdots,x_{l-2})\in(\mathbb{S}\setminus F)^{l-2}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{l-2},x_{l-1})\right\} \\
=\ & \sum_{k=1}^{l-1}\left[\sum_{x_k\in F}\Phi(x_k)\left\{\sum_{(x_1,x_2,\cdots,x_{k-1})\in(\mathbb{S}\setminus F)^{k-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{k-1},x_k)\right\}\right] \\
& + \sum_{x_l\in F}\Phi(x_l)\left\{\sum_{(x_1,x_2,\cdots,x_{l-1})\in(\mathbb{S}\setminus F)^{l-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{l-1},x_l)\right\} \\
& + \sum_{x_l\in\mathbb{S}\setminus F}\Phi(x_l)\left\{\sum_{(x_1,x_2,\cdots,x_{l-1})\in(\mathbb{S}\setminus F)^{l-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{l-1},x_l)\right\} \\
=\ & \sum_{k=1}^{l}\left[\sum_{x_k\in F}\Phi(x_k)\left\{\sum_{(x_1,x_2,\cdots,x_{k-1})\in(\mathbb{S}\setminus F)^{k-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{k-1},x_k)\right\}\right] \\
& + \sum_{x_l\in\mathbb{S}\setminus F}\Phi(x_l)\left\{\sum_{(x_1,x_2,\cdots,x_{l-1})\in(\mathbb{S}\setminus F)^{l-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{l-1},x_l)\right\},
\end{aligned}$$

where the step (a) follows from the property (P1) of the function Φ, and this ends the proof of the bound

(12) for all $n \in \mathbb{N}$. So for any $n \in \mathbb{N}$,

$$\begin{aligned}
\Phi(x_0) &\geq \sum_{k=1}^{n} \left[\sum_{x_k \in F} \underbrace{\Phi(x_k)}_{\geq 1} \left\{ \sum_{(x_1,x_2,\cdots,x_{k-1}) \in (\mathbb{S}\setminus F)^{k-1}} \underbrace{p(x_0,x_1)p(x_1,x_2)\cdots p(x_{k-1},x_k)}_{= \mathbb{P}_{x_0}\{X_1=x_1,\ \cdots,X_{k-1}=x_{k-1},\ X_k=x_k\}} \right\} \right] \\
&\quad + \sum_{x_n \in \mathbb{S}\setminus F} \Phi(x_n) \left\{ \sum_{(x_1,x_2,\cdots,x_{n-1}) \in (\mathbb{S}\setminus F)^{n-1}} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{n-1},x_n) \right\} \\
&\overset{\text{(b)}}{\geq} \sum_{k=1}^{n} \left[\sum_{x_k \in F} \left\{ \sum_{(x_1,x_2,\cdots,x_{k-1}) \in (\mathbb{S}\setminus F)^{k-1}} \mathbb{P}_{x_0}\left\{X_1 = x_1,\ \cdots, X_{k-1} = x_{k-1},\ X_k = x_k\right\} \right\} \right] \qquad (13) \\
&= \sum_{k=1}^{n} \mathbb{P}_{x_0}\left\{X_1 \in \mathbb{S}\setminus F,\ \cdots, X_{k-1} \in \mathbb{S}\setminus F,\ X_k \in F\right\} \\
&\overset{\text{(c)}}{=} \sum_{k=1}^{n} \mathbb{P}_{x_0}\left\{V_F = k\right\} \\
&= \mathbb{P}_{x_0}\left\{V_F \leq n\right\},
\end{aligned}$$

where the above steps (b) and (c) are predicated on the following reasons:

(b) the property (P2) of the function Φ together with the *equation (5.2.3)* in [1];

(c) the definition of the first visiting time V_F to $F \subseteq \mathbb{S}$, and the assumption $x_0 \in \mathbb{S}\setminus F$.

By letting $n \to \infty$ in the inequality (13), we finally obtain $\Phi(x_0) \geq \mathbb{P}_{x_0}\{V_F < +\infty\}$ for every $x_0 \in \mathbb{S}\setminus F$. Furthermore, we know that $\Phi(x) \geq 1 = \mathbb{P}_x\{V_F < +\infty\}$ for all $x \in F$, thereby it establishes our desired claim.

□

According to Claim 1, we know that

$$\varphi(x) = \delta \cdot \Phi(x) \geq \delta \cdot \mathbb{P}_x\left\{V_F < +\infty\right\},\ \forall x \in \mathbb{S}. \qquad (14)$$

Since the superlevel set $\mathcal{L}^+_{\frac{\delta}{2}}(\varphi) = \left\{x \in \mathbb{S} : \varphi(x) \geq \frac{\delta}{2}\right\}$ is finite, its complement $\mathbb{S}\setminus\mathcal{L}^+_{\frac{\delta}{2}}(\varphi)$ is non-empty. Thus, we may take a state $y \in \mathbb{S}$ so that $\varphi(y) < \frac{\delta}{2}$. If follows that

$$\mathbb{P}_y\left\{V_F < +\infty\right\} \leq \frac{1}{\delta} \cdot \varphi(y) < \frac{1}{\delta} \cdot \frac{\delta}{2} = \frac{1}{2}, \qquad (15)$$

and so it's clear that $y \in \mathbb{S}\setminus F$. Now, we assume on the contrary that the Markov chain contains a recurrent state. Due to the irreducibility of the Markov chain, all states are recurrent by *Theorem 5.3.2* in [1]. Also, we find from *Theorem 5.3.2* in [1] that

$$\rho_{yx} = \mathbb{P}_y\left\{T_x < +\infty\right\} \overset{\text{(d)}}{=} \mathbb{P}_y\left\{V_x < +\infty\right\} = 1$$

for every $x \in F$, where the step (d) holds since $y \in \mathbb{S}\setminus F$. As $V_F \leq V_x$ for $x \in F$, we arrive at $\mathbb{P}_y\{V_F < +\infty\} = 1$ and this conclusion violates the inequality (15). Hence, all states in $\mathbb{S}$ of the Markov chain is transient, *i.e.*, the Markov chain $\{X_n\}_{n=0}^{\infty}$ is transient.

Problem 4 (*Exercise 5.3.7.* in [1]).

($\Rightarrow$): Assume that the homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with transition probability $p(\cdot,\cdot):\mathbb{S}\times\mathbb{S}\to[0,1]$ is irreducible and recurrent. Let $f:\mathbb{S}\to\mathbb{R}_+$ be any non-negative superharmonic function with respect to the transition probability $p(\cdot,\cdot):\mathbb{S}\times\mathbb{S}\to[0,1]$. According to the definition of superharmonic functions, the stochastic process $\{f(X_n)\}_{n=0}^{\infty}$ is a non-negative supermartingale with respect to the canonical filtration $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^{\infty}$.

Choose any two states $x, y \in \mathbb{S}$. Since $\{X_n\}_{n=0}^{\infty}$ is an irreducible and recurrent Markov chain, we know that the state y is recurrent and $\rho_{yx} > 0$. By *Theorem 5.3.2* in [1], we obtain

$$\rho_{xy} = \mathbb{P}_x\{T_y < +\infty\} = 1, \forall x, y \in \mathbb{S}. \tag{16}$$

As the time of the first return to state y, $T_y = \inf\{n \geq 1 : X_n = y\}$, is a stopping time with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$, the optional stopping theorem for non-negative supermartingales (*Theorem 4.8.4.* in [1]) yields

$$\begin{aligned} f(x) = \mathbb{E}_x[f(X_0)] &\geq \mathbb{E}_x\left[f\left(X_{T_y}\right)\right] \\ &\overset{\text{(a)}}{=} \mathbb{E}_x\left[f\left(X_{T_y}\right)\mathbb{1}_{\{T_y<+\infty\}}\right] \\ &\overset{\text{(b)}}{=} \mathbb{E}_x\left[f(y)\cdot\mathbb{1}_{\{T_y<+\infty\}}\right] \\ &= f(y)\cdot\mathbb{P}_x\{T_y<+\infty\} \\ &\overset{\text{(c)}}{=} f(y), \end{aligned} \tag{17}$$

where the above steps (a)–(c) can be justified as follows:

(a) the equation (16);

(b) if $T_y < +\infty$, then $X_{T_y} = y$;

(c) the equation (16).

Since the inequality (17) holds for every pair $(x, y) \in \mathbb{S}\times\mathbb{S}$ of states, we may conclude that the function f is constant on $\mathbb{S}$.

($\Leftarrow$): Conversely, we now assume that the homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with transition probability $p(\cdot,\cdot):\mathbb{S}\times\mathbb{S}\to[0,1]$ is irreducible and every non-negative superharmonic function with respect to the transition probability $p(\cdot,\cdot):\mathbb{S}\times\mathbb{S}\to[0,1]$ is constant. Given any fixed state $y\in\mathbb{S}$, we consider the first visiting time $V_y := \inf\{n \geq 0 : X_n = y\}$ to state y. Define a function $h:\mathbb{S}\to\mathbb{R}_+$ by

$$h(x) := \mathbb{P}_x\{V_y < +\infty\}, \ \forall x \in \mathbb{S}.$$

It's clear from the definition of $h:\mathbb{S}\to\mathbb{R}_+$ that $h(y) = 1$. We claim that $h:\mathbb{S}\to\mathbb{R}_+$ is a superharmonic function with respect to the transition probability $p(\cdot,\cdot)$. In order to prove this claim, we define a function $Y:\Omega_0\to\mathbb{R}$ by

$$Y(\omega) := \mathbb{1}_{\{V_y<+\infty\}}(\omega) = \mathbb{1}_{\{X_n=y \text{ for some } y\geq 0\}}(\omega), \ \forall \omega \in \Omega_0.$$

It's clear that Y is a bounded measurable function on the sequence space $(\Omega_0, \mathcal{F}_\infty)$. Also, one can see that

$$\begin{aligned}(Y \circ \theta_1)(\omega) &= \mathbb{1}_{\{X_n = y \text{ for some } y \geq 0\}}(\theta_1(\omega)) \\ &= \mathbb{1}_{\{X_n = y \text{ for some } y \geq 1\}}(\omega) \\ &\leq \mathbb{1}_{\{X_n = y \text{ for some } y \geq 0\}}(\omega) = Y(\omega)\end{aligned} \tag{18}$$

for every $\omega \in \Omega_0$. Thus,

$$\begin{aligned}h(x) &= \mathbb{E}_x\left[\mathbb{1}_{\{V_y < +\infty\}}\right] = \mathbb{E}_x[Y] \\ &\overset{\text{(d)}}{\geq} \mathbb{E}_x[Y \circ \theta_1] \\ &= \mathbb{E}_x\left[\mathbb{E}_x[Y \circ \theta_1 | \mathcal{F}_1]\right] \\ &\overset{\text{(e)}}{=} \mathbb{E}_x\left[\mathbb{E}_{X_1}[Y]\right] \\ &= \mathbb{E}_x\left[\sum_{z \in \mathbb{S}} \left(\mathbb{E}_{X_1}[Y] \cdot \mathbb{1}_{\{X_1 = z\}}\right)\right] \\ &\overset{\text{(f)}}{=} \sum_{z \in \mathbb{S}} \mathbb{E}_x\left[\mathbb{E}_{X_1}[Y] \cdot \mathbb{1}_{\{X_1 = z\}}\right] \\ &= \sum_{z \in \mathbb{S}} \mathbb{E}_x\left[\mathbb{E}_z[Y] \cdot \mathbb{1}_{\{X_1 = z\}}\right] \\ &= \sum_{z \in \mathbb{S}} \mathbb{P}_x\{X_1 = z\} \mathbb{E}_z[Y] \\ &= \sum_{z \in \mathbb{S}} p(x, z) h(z),\end{aligned} \tag{19}$$

thereby $h : \mathbb{S} \to \mathbb{R}_+$ is a non-negative superharmonic function with respect to the transition probability $p(\cdot, \cdot)$. Here, the above steps (d)–(f) can be verified as follows:

(d) it is simply the inequality (18);

(e) the Markov property (*Theorem 5.2.3* in [1]);

(f) the Fubini-Tonelli's theorem, since the summands are non-negative.

Hence, h is a constant function on $\mathbb{S}$. Since we already know that $h(y) = 1$, we can conclude that $h(x) = 1$ for all $x \in \mathbb{S}$. As the fixed state $y \in \mathbb{S}$ is arbitrarily chosen, we obtain

$$\mathbb{P}_x\{V_y < +\infty\} = 1 \tag{20}$$

for all $x, y \in \mathbb{S}$.

Finally, we confirm that the Markov chain $\{X_n\}_{n=0}^\infty$ is recurrent. Fix any state $x \in \mathbb{S}$ and choose another state $y \in \mathbb{S} \setminus \{x\}$ arbitrarily. Due to (20), we have $\rho_{xy} = \mathbb{P}_x\{V_y < +\infty\} = 1$ and $\rho_{yx} = \mathbb{P}_y\{V_x < +\infty\} = 1$. Applying Problem 2 (*Exercise 5.3.2* in [1]) gives $\rho_{xx} \geq \rho_{xy} \cdot \rho_{yx} = 1$, so $\rho_{xx} = 1$ for all $x \in \mathbb{S}$. Hence, all states in $\mathbb{S}$ are recurrent, *i.e.*, the transition probability $p(\cdot, \cdot)$ and the Markov chain $\{X_n\}_{n=0}^\infty$ are recurrent.

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.