MAS651 Theory of Stochastic Processes
Homework #2

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$. Also, we use the symbol $\mathbb{S}$ instead of S to denote the underlying state space of stochastic processes.

Problem 1 (*Exercise 5.2.1.* in [1]).

To begin with, we introduce some notations: $\mathcal{F}_n := \sigma(\{X_k : k \in [0:n]\})$ and $\mathcal{G}_n := \sigma(\{X_k : k \geq n\})$ for $n \in \mathbb{Z}_+$. Given a fixed time step $n \in \mathbb{Z}_+$, and two events $A \in \mathcal{F}_n$ and $B \in \mathcal{G}_n$, we have $\mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B$, we obtain

$$\begin{aligned}
\mathbb{P}_\mu\{A \cap B | X_n\} &= \mathbb{E}_\mu[\mathbb{1}_{A\cap B} | \sigma(X_n)] \\
&= \mathbb{E}_\mu[\mathbb{1}_A \mathbb{1}_B | \sigma(X_n)] \\
&\overset{\text{(a)}}{=} \mathbb{E}_\mu[\mathbb{E}_\mu[\mathbb{1}_A \mathbb{1}_B | \mathcal{F}_n] | \sigma(X_n)] \\
&\overset{\text{(b)}}{=} \mathbb{E}_\mu[\mathbb{1}_A \cdot \mathbb{E}_\mu[\mathbb{1}_B | \mathcal{F}_n] | \sigma(X_n)]
\end{aligned} \tag{1}$$

$\mathbb{P}_\mu$-almost surely, where the step (a) follows from *Theorem 4.1.13* in [1], and the step (b) holds by *Theorem 4.1.14* in [1] together with the fact $A \in \mathcal{F}_n$.

Claim 1. *For any $B \in \mathcal{G}_n$, $\mathbb{E}_\mu[\mathbb{1}_B | \mathcal{F}_n]$ is $\sigma(X_n)$-measurable.*

Proof of Claim 1.

Let $\mathcal{P}_k := \{\{\omega \in \Omega_0 : \omega_n \in A_0, \omega_{n+1} \in A_1, \cdots, \omega_{n+k} \in A_k\} : A_0, A_1, \cdots, A_k \in \mathcal{S}\}$, $k \in \mathbb{Z}_+$, where $(\mathbb{S}, \mathcal{S})$ is the underlying nice state space and $(\Omega_0, \mathcal{F}_\infty)$ denotes the sequence space constructed from $(\mathbb{S}, \mathcal{S})$. Here, we recall that a measurable space $(\mathbb{S}, \mathcal{S})$ is *nice* (or said to be a *standard Borel space*) if there is a bijection $\varphi : \mathbb{S} \to \mathbb{R}$ such that both $\varphi : (\mathbb{S}, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\varphi^{-1} : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{S}, \mathcal{S})$ are measurable, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field on $\mathbb{R}$ (See *Section 2.1* of [1] for further details). Set $\mathcal{P} := \bigcup_{k=0}^\infty \mathcal{P}_k$. Choose any two elements $C, D \in \mathcal{P}$. Then, $C \in \mathcal{P}_k$ and $D \in \mathcal{P}_l$ for some $k, l \in \mathbb{Z}_+$. We may write C and D by

$$\begin{aligned}
C &= \{\omega \in \Omega_0 : \omega_n \in A_0, \omega_{n+1} \in A_1, \cdots, \omega_{n+k} \in A_k\}; \\
D &= \{\omega \in \Omega_0 : \omega_n \in B_0, \omega_{n+1} \in B_1, \cdots, \omega_{n+l} \in B_l\},
\end{aligned}$$

for some $A_0, A_1, \cdots, A_k, B_0, B_1, \cdots, B_l \in \mathcal{S}$. Assuming $k \le l$, we have

$$C \cap D = \{\omega \in \Omega_0 : \omega_n \in A_0 \cap B_0, \cdots, \omega_{n+k} \in A_k \cap B_k, \omega_{n+k+1} \in B_{k+1}, \cdots, \omega_{n+l} \in B_l\} \in \mathcal{P}_l,$$

thereby $C \cap D \in \mathcal{P}$. Thus, $\mathcal{P}$ is a π-system on Ω_0 with $\mathcal{G}_n = \sigma(\mathcal{P})$.

Now, choose any $E \in \mathcal{P}$. Then, it can be written as

$$E = \{\omega \in \Omega_0 : \omega_n \in A_0, \omega_{n+1} \in A_1, \cdots, \omega_{n+k} \in A_k\}$$

for some $k \in \mathbb{Z}_+$ and $A_0, A_1, \cdots, A_k \in \mathcal{S}$. Define a function $Y : \Omega_0 \to \mathbb{R}$ by

$$Y(\omega) := \prod_{j=0}^{k} \mathbb{1}_{A_j}(\omega_j), \ \omega \in \Omega_0.$$

It's clear that Y is a bounded measurable function from the sequence space $(\Omega_0, \mathcal{F}_\infty)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Moreover, one can see that

$$\mathbb{1}_E(\omega) = \prod_{j=0}^{k} \mathbb{1}_{A_j}(\omega_{n+j}) = \prod_{j=0}^{k} \mathbb{1}_{A_j}\left([\theta_n(\omega)]_j\right) = (Y \circ \theta_n)(\omega) \tag{2}$$

for all $\omega \in \Omega_0$. Therefore,

$$\mathbb{E}_\mu [\mathbb{1}_E | \mathcal{F}_n] \overset{\text{(c)}}{=} \mathbb{E}_\mu [Y \circ \theta_n | \mathcal{F}_n] \overset{\text{(d)}}{=} \mathbb{E}_{X_n}[Y] \tag{3}$$

$\mathbb{P}_\mu$-almost surely, where the step (c) is due to (2), and the step (d) follows from the Markov property. Since the last term of (3) is $\sigma(X_n)$-measurable, we conclude that $\mathbb{E}_\mu [\mathbb{1}_E | \mathcal{F}_n]$ is $\sigma(X_n)$-measurable for all $E \in \mathcal{P}$.

As the final step, we show that $\mathbb{E}_\mu [\mathbb{1}_B | \mathcal{F}_n]$ is $\sigma(X_n)$-measurable for all $B \in \mathcal{G}_n$. Let

$$\mathcal{L} := \{B \in \mathcal{G}_n : \mathbb{E}_\mu [\mathbb{1}_B | \mathcal{F}_n] \text{ is } \sigma(X_n)\text{-measurable.}\}.$$

We have already seen that $\mathcal{P} \subseteq \mathcal{L}$ in the previous paragraph. Now, we claim that $\mathcal{L}$ is a λ-system on Ω_0.

1. It's clear that $\varnothing$ and Ω_0 belongs to $\mathcal{L}$;

2. Assume that $A, B \in \mathcal{L}$. Then,

$$\mathbb{E}_\mu \left[\mathbb{1}_{B \setminus A} \middle| \mathcal{F}_n\right] \overset{\mathbb{P}_\mu\text{-a.s.}}{=} \mathbb{E}_\mu [\mathbb{1}_B | \mathcal{F}_n] - \mathbb{E}_\mu [\mathbb{1}_A | \mathcal{F}_n]$$

   is also $\sigma(X_n)$-measurable, thereby $B \setminus A \in \mathcal{L}$;

3. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{L}$ with $A_k \uparrow A$ as $k \to \infty$. From $\mathbb{1}_{A_k} \uparrow \mathbb{1}_A$ as $k \to \infty$, we obtain from the monotone convergence theorem for conditional expectations (*Theorem 4.1.9-(c)* in [1]) that

$$\mathbb{E}_\mu [\mathbb{1}_{A_k} | \mathcal{F}_n] \uparrow \mathbb{E}_\mu [\mathbb{1}_A | \mathcal{F}_n]$$

   $\mathbb{P}_\mu$-almost surely, as $k \to \infty$. Since each $\mathbb{E}_\mu [\mathbb{1}_{A_k} | \mathcal{F}_n]$ is $\sigma(X_n)$-measurable, $\mathbb{E}_\mu [\mathbb{1}_A | \mathcal{F}_n]$ is also $\sigma(X_n)$-measurable.

From the above observations, $\mathcal{L}$ is a λ-system on Ω_0. Due to the π-λ theorem (*Theorem 2.1.6* in [1]), we get $\mathcal{L} = \sigma(\mathcal{P}) = \mathcal{G}_n$. This completes the proof of Claim 1.

□

According to Claim 1, we know that $\mathbb{E}_\mu\left[\mathbb{1}_B \middle| \mathcal{F}_n\right]$ is $\sigma(X_n)$-measurable. Thus,

$$\mathbb{E}_\mu\left[\mathbb{1}_B \middle| \mathcal{F}_n\right] \overset{\mathbb{P}_\mu\text{-a.s.}}{=} \mathbb{E}_\mu\left[\mathbb{1}_B \middle| \sigma(X_n)\right] \tag{4}$$

by *Theorem 4.1.12* in [1]. Plugging (4) into (1) yields

$$\begin{aligned}
\mathbb{P}_\mu\left\{A \cap B \middle| X_n\right\} &= \mathbb{E}_\mu\left[\mathbb{1}_A \cdot \mathbb{E}_\mu\left[\mathbb{1}_B \middle| \mathcal{F}_n\right] \middle| \sigma(X_n)\right] \\
&= \mathbb{E}_\mu\left[\mathbb{1}_A \cdot \mathbb{E}_\mu\left[\mathbb{1}_B \middle| \sigma(X_n)\right] \middle| \sigma(X_n)\right] \\
&\overset{\text{(e)}}{=} \mathbb{E}_\mu\left[\mathbb{1}_A \middle| \sigma(X_n)\right] \mathbb{E}_\mu\left[\mathbb{1}_B \middle| \sigma(X_n)\right] \\
&= \mathbb{P}_\mu\left\{A \middle| X_n\right\} \mathbb{P}_\mu\left\{B \middle| X_n\right\}
\end{aligned}$$

$\mathbb{P}_\mu$-almost surely, where the step (e) is due to *Theorem 4.1.14* in [1], and we are done.

Hereafter, we assume throughout the rest of homework problems that the underlying state space $\mathbb{S}$ is countable and it is equipped with the discrete σ-field $2^{\mathbb{S}}$. Since $\left(\mathbb{S}, 2^{\mathbb{S}}\right)$ is a *nice measurable space*, it admits the canonical construction in *Section 5.2* in [1] of the probability measure $\mathbb{P}_\mu$ on the sequence space $(\Omega_0, \mathcal{F}_\infty)$ so that the sequence of coordinate maps $\{X_n(\omega) := \omega_n\}_{n=0}^\infty$ is a homogeneous Markov chain with initial distribution μ and transition probability $p(\cdot,\cdot) : \mathbb{S} \times 2^{\mathbb{S}} \to [0,1]$. We remark that it is conventional to write $p(x,y) := p(x,\{y\})$ for $x, y \in \mathbb{S}$.

Problem 2 (*Exercise 5.2.4.* in [1]: First entrance decomposition)**.**

We begin the proof by defining the functions $Y_m : \Omega_0 \to \mathbb{R}$, $m \in \mathbb{Z}_+$, by

$$Y_m(\omega) := \mathbb{1}_{\{X_{n-m}=y\}}(\omega) = \mathbb{1}_{\{\omega_{n-m}=y\}},\ \forall \omega \in \Omega_0,$$

if $0 \le m \le n$, and $Y_m(\omega) := 0$ for all $\omega \in \Omega_0$ otherwise. It's clear that all Y_m are bounded and measurable functions on the sequence space $(\Omega_0, \mathcal{F}_\infty)$. Also, according to the definition of the *hitting time* T_y, we have

$$\{T_y = n\} = \{X_1 \in \mathbb{S} \setminus \{y\}, \cdots, X_{n-1} \in \mathbb{S} \setminus \{y\}, X_n = y\} \in \mathcal{F}_n$$

for every $n \in \mathbb{Z}_+$, where $\{\mathcal{F}_n := \sigma\left(\{X_k : k \in [0:n]\}\right)\}_{n=0}^\infty$ denotes the canonical filtration of $\{X_n\}_{n=0}^\infty$. Thus, T_y is a stopping time with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. Moreover, one can see that if $T_y(\omega) < +\infty$, then

$$\left(Y_{T_y} \circ \theta_{T_y}\right)(\omega) = \mathbb{1}_{\{X_n=y\}}(\omega),\ \forall \omega \in \Omega_0. \tag{5}$$

Hence, we have

$$\begin{aligned}
p^n(x,y) &= \mathbb{P}_x\{X_n = y\} = \mathbb{E}_x\left[\mathbb{1}_{\{X_n=y\}}\right] \\
&\overset{\text{(a)}}{=} \mathbb{E}_x\left[\mathbb{1}_{\{X_n=y\}} \cdot \mathbb{1}_{\{T_y<+\infty\}}\right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_x\left[\left(Y_{T_y} \circ \theta_{T_y}\right) \cdot \mathbb{1}_{\{T_y<+\infty\}}\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\left(Y_{T_y} \circ \theta_{T_y}\right) \cdot \mathbb{1}_{\{T_y<+\infty\}} \middle| \mathcal{F}_{T_y}\right]\right] \\
&\overset{\text{(c)}}{=} \mathbb{E}_x\left[\mathbb{E}_x\left[\left(Y_{T_y} \circ \theta_{T_y}\right) \cdot \middle| \mathcal{F}_{T_y}\right] \mathbb{1}_{\{T_y<+\infty\}}\right] \\
&\overset{\text{(d)}}{=} \mathbb{E}_x\left[\mathbb{E}_{X_{T_y}}\left[Y_{T_y}\right] \cdot \mathbb{1}_{\{T_y<+\infty\}}\right] \\
&\overset{\text{(e)}}{=} \mathbb{E}_x\left[\mathbb{E}_y\left[Y_{T_y}\right] \cdot \mathbb{1}_{\{T_y<+\infty\}}\right] \\
&= \mathbb{E}_x\left[\sum_{m=0}^{\infty} \mathbb{E}_y\left[Y_{T_y}\right] \cdot \mathbb{1}_{\{T_y=m\}}\right] \\
&\overset{\text{(f)}}{=} \sum_{m=0}^{\infty} \mathbb{E}_x\left[\mathbb{E}_y\left[Y_{T_y}\right] \cdot \mathbb{1}_{\{T_y=m\}}\right] \\
&= \sum_{m=0}^{\infty} \mathbb{E}_y\left[Y_m\right] \cdot \mathbb{E}_x\left[\mathbb{1}_{\{T_y=m\}}\right] \\
&= \sum_{m=0}^{\infty} \mathbb{P}_x\{T_y = m\} \underbrace{\mathbb{P}_y\{X_{n-m} = y\}}_{=\, p^{n-m}(y,y)}.
\end{aligned}$$

Here, the above steps (a)–(f) can be justified as follows:

(a) $\{X_n = y\} \subseteq \{T_y < +\infty\}$;

(b) the equality (5);

(c) $\{T_y < +\infty\} \in \mathcal{F}_{T_y}$. To see this, we notice that

$$\{T_y < +\infty\} \cap \{T_y = n\} = \{T_y = n\} \in \mathcal{F}_n, \ \forall n \in \mathbb{Z}_+,$$

since T_y is a stopping time with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$;

(d) the strong Markov property (*Theorem 5.2.5* in [1]);

(e) On the event $\{T_y < +\infty\}$, we have $X_{T_y} = y$;

(f) the Fubini-Tonelli's theorem, since the summands are non-negative.

This establishes the desired result.

Problem 3 (*Exercise 5.2.6.* in [1]**).**

Since $\mathbb{S} \setminus C$ is finite, $\inf\{\mathbb{P}_x\{T_C < +\infty\} : x \in \mathbb{S} \setminus C\} > 0$. Take $\epsilon := \frac{1}{2}\inf\{\mathbb{P}_x\{T_C < +\infty\} : x \in \mathbb{S} \setminus C\} \in (0,1)$. As $\mathbb{P}_x\{T_C < +\infty\} = \lim_{n\to\infty} \uparrow \mathbb{P}_x\{T_C \le n\} > \epsilon$ for every $x \in \mathbb{S} \setminus C$, there exists a positive integer $N(x) \in \mathbb{N}$ such that $\mathbb{P}_x\{T_C \le n\} \ge \epsilon$ for every $n \ge N(x)$. Let $N := \max\{N(x) : x \in \mathbb{S} \setminus C\} \in \mathbb{N}$. Then, we have

$$\mathbb{P}_x\{T_C > N\} = 1 - \mathbb{P}_x\{T_C \le N\} \le 1 - \epsilon \tag{6}$$

for every $x \in \mathbb{S} \setminus C$, since $N \geq N(x)$ for all $x \in \mathbb{S} \setminus C$.

Now, define $Y : \Omega_0 \to \mathbb{R}$ by

$$Y(\omega) := \mathbb{1}_{\{T_C > N\}}(\omega) = \mathbb{1}_{\bigcap_{i=1}^{N}\{X_i \in \mathbb{S}\setminus C\}}(\omega), \ \forall \omega \in \Omega_0.$$

Then, Y is clearly a bounded, measurable function from the sequence space $(\Omega_0, \mathcal{F}_\infty)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Moreover, we can see that for every $k \geq 2$,

$$\left(Y \circ \theta_{(k-1)N}\right)(\omega) = \mathbb{1}_{\bigcap_{i=(k-1)N+1}^{kN}\{X_i \in \mathbb{S}\setminus C\}}(\omega), \forall \omega \in \Omega_0,$$

thereby we get

$$\begin{aligned}
\left(Y \circ \theta_{(k-1)N}\right)(\omega) \cdot \mathbb{1}_{\{T_C > (k-1)N\}}(\omega) &= \mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N+1} \in \mathbb{S}\setminus C,\ X_{(k-1)N+2} \in \mathbb{S}\setminus C,\ \cdots,\ X_{kN} \in \mathbb{S}\setminus C\right\}}(\omega) \\
&= \mathbb{1}_{\bigcap_{i=1}^{kN}\{X_i \in \mathbb{S}\setminus C\}}(\omega) \\
&= \mathbb{1}_{\{T_C > kN\}}(\omega)
\end{aligned} \tag{7}$$

for all $\omega \in \Omega_0$. Therefore, we obtain for every $y \in \mathbb{S} \setminus C$ and $k \geq 2$ that

$$\begin{aligned}
\mathbb{P}_y\{T_C > kN\} &= \mathbb{E}_y\left[\mathbb{1}_{\{T_C > kN\}}\right] \\
&\overset{\text{(a)}}{=} \mathbb{E}_y\left[\left(Y \circ \theta_{(k-1)N}\right) \cdot \mathbb{1}_{\{T_C > (k-1)N\}}\right] \\
&= \mathbb{E}_y\left[\mathbb{E}_y\left[\left(Y \circ \theta_{(k-1)N}\right) \cdot \mathbb{1}_{\{T_C > (k-1)N\}} \middle| \mathcal{F}_{(k-1)N}\right]\right] \\
&\overset{\text{(b)}}{=} \mathbb{E}_y\left[\mathbb{E}_y\left[Y \circ \theta_{(k-1)N} \middle| \mathcal{F}_{(k-1)N}\right] \mathbb{1}_{\{T_C > (k-1)N\}}\right] \\
&\overset{\text{(c)}}{=} \mathbb{E}_y\left[\mathbb{E}_{X_{(k-1)N}}[Y] \cdot \mathbb{1}_{\{T_C > (k-1)N\}}\right] \\
&\overset{\text{(d)}}{=} \mathbb{E}_y\left[\mathbb{E}_{X_{(k-1)N}}[Y] \cdot \mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} \in \mathbb{S}\setminus C\right\}}\right] \\
&= \mathbb{E}_y\left[\mathbb{E}_{X_{(k-1)N}}[Y]\left(\sum_{x \in \mathbb{S}\setminus C} \mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} = x\right\}}\right)\right] \\
&\overset{\text{(e)}}{=} \sum_{x \in \mathbb{S}\setminus C} \mathbb{E}_y\left[\mathbb{E}_{X_{(k-1)N}}[Y] \cdot \mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} = x\right\}}\right] \\
&= \sum_{x \in \mathbb{S}\setminus C} \mathbb{E}_y\left[\mathbb{E}_x[Y] \cdot \mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} = x\right\}}\right] \\
&= \sum_{x \in \mathbb{S}\setminus C} \mathbb{E}_y\left[\mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} = x\right\}}\right] \cdot \mathbb{P}_x\{T_C > N\} \\
&\overset{\text{(f)}}{\leq} (1-\epsilon) \sum_{x \in \mathbb{S}\setminus C} \mathbb{E}_y\left[\mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} = x\right\}}\right] \\
&\overset{\text{(g)}}{=} (1-\epsilon)\mathbb{E}_y\left[\sum_{x \in \mathbb{S}\setminus C} \mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} = x\right\}}\right] \\
&= (1-\epsilon)\mathbb{E}_y\left[\mathbb{1}_{\left\{T_C > (k-1)N,\ X_{(k-1)N} \in \mathbb{S}\setminus C\right\}}\right] \\
&\overset{\text{(h)}}{=} (1-\epsilon)\mathbb{E}_y\left[\mathbb{1}_{\{T_C > (k-1)N\}}\right] \\
&= (1-\epsilon)\mathbb{P}_y\{T_C > (k-1)N\},
\end{aligned}$$

Here, the above steps (a)–(h) can be justified as follows:

(a) the equality (7);

(b) $\{T_C > (k-1)N\} = \bigcap_{i=1}^{(k-1)N} \{X_i \in \mathbb{S} \setminus C\} \in \mathcal{F}_{(k-1)N}$;

(c) the Markov property (*Theorem 5.2.3* in [1]);

(d) $\{T_C > (k-1)N\} \subseteq \left\{X_{(k-1)N} \in \mathbb{S} \setminus C\right\}$;

(e) we can change the order between expectation and summation since $\mathbb{S} \setminus C$ is finite;

(f) the bound (6);

(g) we can change the order between expectation and summation since $\mathbb{S} \setminus C$ is finite;

(h) $\{T_C > (k-1)N\} \subseteq \left\{X_{(k-1)N} \in \mathbb{S} \setminus C\right\}$.

We remark that $\{\mathcal{F}_n\}_{n=0}^{\infty}$ denotes the canonical filtration of the given Markov chain $\{X_n\}_{n=0}^{\infty}$, *i.e.*, $\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)$ for every $n \in \mathbb{Z}_+$. Hence, we can deduce inductively that

$$\mathbb{P}_y \{T_C > kN\} \leq (1-\epsilon)^{k-1} \cdot \mathbb{P}_y \{T_C > N\} \overset{\text{(i)}}{\leq} (1-\epsilon)^k$$

for every $k \in \mathbb{N}$ and $y \in \mathbb{S} \setminus C$, where the step (i) is simply the bound (6). This completes the proof of the desired result.

Problem 4 (*Exercise 5.2.7.* in [1]: Exit distributions)**.**

(i) To begin with, we can see for every $C \subseteq \mathbb{S}$ that the *first visiting time to C*, V_C, is a stopping time with respect to the canonical filtration $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^{\infty}$. To see this, we notice that

$$\{V_C = n\} = \{X_0 \in \mathbb{S} \setminus C, \cdots, X_{n-1} \in \mathbb{S} \setminus C, X_n \in C\} \in \mathcal{F}_n, \ \forall n \in \mathbb{Z}_+.$$

Now, we define a function $Y : \Omega_0 \to \mathbb{R}$ by

$$Y(\omega) := \mathbb{1}_{\{V_A < V_B\}}(\omega) = \begin{cases} 1 & \text{if } \inf\{n \geq 0 : X_n \in A\} < \inf\{n \geq 0 : X_n \in B\}; \\ 0 & \text{otherwise.} \end{cases}$$

Since both V_A and V_B are stopping times with respect to $\{\mathcal{F}_n\}_{n=0}^{\infty}$ defined on the sequence space $(\Omega_0, \mathcal{F}_\infty)$, $\{V_A < V_B\} \in \mathcal{F}_\infty$. Thus, $Y : (\Omega_0, \mathcal{F}_\infty) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a bounded, measurable function, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field on $\mathbb{R}$. Moreover, we may observe that if $X_0 \in \mathbb{S} \setminus (A \cup B)$, then

$$\begin{aligned} (Y \circ \theta_1)(\omega) &= \begin{cases} 1 & \text{if } \inf\{n \geq 1 : X_n \in A\} < \inf\{n \geq 1 : X_n \in B\}; \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1 & \text{if } \inf\{n \geq 0 : X_n \in A\} < \inf\{n \geq 0 : X_n \in B\}; \\ 0 & \text{otherwise.} \end{cases} \\ &= Y(\omega) \end{aligned} \tag{8}$$

for all $\omega \in \Omega_0$. Hence, the following holds: for every $x \in \mathbb{S} \setminus (A \cup B)$,

$$
\begin{aligned}
h(x) &= \mathbb{P}_x \{V_A < V_B\} = \mathbb{E}_x [Y] \\
&\overset{\text{(a)}}{=} \mathbb{E}_x [Y \circ \theta_1] \\
&= \mathbb{E}_x [\mathbb{E}_x [Y \circ \theta_1 | \mathcal{F}_1]] \\
&\overset{(b)}{=} \mathbb{E}_x [\mathbb{E}_{X_1} [Y]] \\
&= \mathbb{E}_x \left[\sum_{y \in \mathbb{S}} \mathbb{E}_{X_1} [Y] \cdot \mathbb{1}_{\{X_1 = y\}} \right] \\
&\overset{\text{(c)}}{=} \sum_{y \in \mathbb{S}} \mathbb{E}_x \left[\mathbb{E}_{X_1} [Y] \cdot \mathbb{1}_{\{X_1 = y\}} \right] \\
&= \sum_{y \in \mathbb{S}} \mathbb{E}_x \left[\mathbb{E}_y [Y] \cdot \mathbb{1}_{\{X_1 = y\}} \right] \\
&= \sum_{y \in \mathbb{S}} \underbrace{\mathbb{P}_x \{X_1 = y\}}_{= p(x,y)} \cdot \underbrace{\mathbb{P}_y \{V_A < V_B\}}_{= h(y)} \\
&= \sum_{y \in \mathbb{S}} p(x,y) h(y),
\end{aligned}
$$

Here, the above steps (a)–(c) can be verified as follows:

(a) it follows from (8) together with the assumption $x \in \mathbb{S} \setminus (A \cup B)$;

(b) the Markov property (*Theorem 5.2.3* in [1]);

(c) the Fubini-Tonelli's theorem, since the summands are non-negative.

(ii) Let μ denote the initial distribution of the Markov chain $\{X_n\}_{n=0}^{\infty}$, $h : \mathbb{S} \to \mathbb{R}$ be any bounded function satisfying the given condition $(*)$, and $M_n := h\left(X_{n \wedge V_{A \cup B}}\right)$ for $n \in \mathbb{Z}_+$. Then, one can see that for $n \in \mathbb{N}$,

$$
\begin{aligned}
M_n &= h(X_n) \mathbb{1}_{\{V_{A \cup B} \geq n\}} + h\left(X_{V_{A \cup B}}\right) \mathbb{1}_{\{V_{A \cup B} < n\}} \\
&= h(X_n) \mathbb{1}_{\{V_{A \cup B} \geq n\}} + \sum_{k=0}^{n-1} h\left(X_{V_{A \cup B}}\right) \mathbb{1}_{\{V_{A \cup B} = k\}}.
\end{aligned} \tag{9}
$$

It's clear that $M_n \in L^1(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_\mu)$, *i.e.*, M_n is $\mathbb{P}_\mu$-integrable as h is bounded. From (9), we reach

$$
\begin{aligned}
\mathbb{E}_\mu [M_n | \mathcal{F}_{n-1}] &= \mathbb{E}_\mu \left[h(X_n) \mathbb{1}_{\{V_{A \cup B} \geq n\}} \middle| \mathcal{F}_{n-1} \right] + \sum_{k=0}^{n-1} \mathbb{E}_\mu \left[h\left(X_{V_{A \cup B}}\right) \mathbb{1}_{\{V_{A \cup B} = k\}} \middle| \mathcal{F}_{n-1} \right] \\
&= \mathbb{E}_\mu \left[h(X_n) \mathbb{1}_{\{V_{A \cup B} \geq n\}} \middle| \mathcal{F}_{n-1} \right] + \sum_{k=0}^{n-1} \mathbb{E}_\mu \left[h(X_k) \mathbb{1}_{\{V_{A \cup B} = k\}} \middle| \mathcal{F}_{n-1} \right] \\
&\overset{\text{(d)}}{=} \mathbb{E}_\mu \left[h(X_n) \mathbb{1}_{\{V_{A \cup B} \geq n\}} \middle| \mathcal{F}_{n-1} \right] + \sum_{k=0}^{n-1} h(X_k) \mathbb{1}_{\{V_{A \cup B} = k\}},
\end{aligned} \tag{10}
$$

$\mathbb{P}_\mu$-almost surely, where the step (a) follows from the fact $h(X_k) \mathbb{1}_{\{V_{A \cup B} = k\}} \in \mathcal{F}_k \subseteq \mathcal{F}_{n-1}$ for every $k \in [0 : n-1]$, which holds since $V_{A \cup B}$ is a stopping time with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$. At this point, we claim the following statement.

Claim 2. $\mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B}\geq n\}}\middle|\mathcal{F}_{n-1}\right] \overset{\mathbb{P}_\mu\text{-a.s.}}{=} h(X_{n-1})\mathbb{1}_{\{V_{A\cup B}\geq n\}}$.

Proof of Claim 2.

To begin with, we note from $\{V_{A\cup B}\geq k\}=\{X_0\in\mathbb{S}\setminus(A\cup B), X_1\in\mathbb{S}\setminus(A\cup B),\cdots,X_{k-1}\in\mathbb{S}\setminus(A\cup B)\}$ that

$$\mathbb{1}_{\{V_{A\cup B}\geq k\}}(\omega)=\prod_{j=0}^{k-1}\mathbb{1}_{\mathbb{S}\setminus(A\cup B)}(\omega_j) \tag{11}$$

for all $\omega\in\Omega_0$ and $k\in\mathbb{N}$. Let $\mathcal{P}_k:=\left\{\{\omega\in\Omega_0:\omega_0\in A_0,\omega_1\in A_1,\cdots,\omega_k\in A_k\}:A_0,A_1,\cdots,A_k\in\mathcal{S}=2^{\mathbb{S}}\right\}$ for $k\in\mathbb{Z}_+$. Then, $\mathcal{P}_k$ is a π-system on Ω_0 with $\mathcal{F}_k=\sigma(X_0,X_1,\cdots,X_k)=\sigma(\mathcal{P}_k)$. Firstly, we claim that

$$\mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B}\geq n\}}\cdot\mathbb{1}_E\right]=\mathbb{E}_\mu\left[h(X_{n-1})\mathbb{1}_{\{V_{A\cup B}\geq n\}}\cdot\mathbb{1}_E\right] \tag{12}$$

for all $E\in\mathcal{P}_{n-1}$. Given any $E\in\mathcal{P}_{n-1}$, it can be written by

$$E=\{X_0\in A_0,X_1\in A_1,\cdots,X_{n-1}\in A_{n-1}\}$$

for some $A_0,A_1,\cdots,A_{n-1}\in\mathcal{S}=2^{\mathbb{S}}$. Therefore,

$$\begin{aligned}
\mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B}\geq n\}}\cdot\mathbb{1}_E\right] &\overset{\text{(e)}}{=} \mathbb{E}_\mu\left[\prod_{k=0}^{n-1}\left(\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_k}(X_k)\right)h(X_n)\right] \\
&\overset{\text{(f)}}{=} \int_{\mathbb{S}}\mu(\mathrm{d}x_0)\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_0}(x_0)\left[\int_{\mathbb{S}}p(x_0,\mathrm{d}x_1)\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_1}(x_1)\right.\\
&\quad\left.\left[\cdots\left[\int_{\mathbb{S}}p(x_{n-1},\mathrm{d}x_n)h(x_n)\right]\cdots\right]\right] \\
&= \sum_{x_0\in\{\mathbb{S}\setminus(A\cup B)\}\cap A_0}\mu(x_0)\left[\sum_{x_1\in\{\mathbb{S}\setminus(A\cup B)\}\cap A_1}p(x_0,x_1)\right.\\
&\quad\left.\left[\cdots\left[\sum_{x_{n-1}\in\{\mathbb{S}\setminus(A\cup B)\}\cap A_{n-1}}p(x_{n-2},x_{n-1})\underbrace{\left[\sum_{x_n\in\mathbb{S}}p(x_{n-1},x_n)h(x_n)\right]}_{=\,h(x_{n-1})}\right]\cdots\right]\right] \qquad (13)\\
&\overset{\text{(g)}}{=} \sum_{x_0\in\{\mathbb{S}\setminus(A\cup B)\}\cap A_0}\mu(x_0)\left[\sum_{x_1\in\{\mathbb{S}\setminus(A\cup B)\}\cap A_1}p(x_0,x_1)\right.\\
&\quad\left.\left[\cdots\left[\sum_{x_{n-1}\in\{\mathbb{S}\setminus(A\cup B)\}\cap A_{n-1}}p(x_{n-2},x_{n-1})h(x_{n-1})\right]\cdots\right]\right] \\
&= \int_{\mathbb{S}}\mu(\mathrm{d}x_0)\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_0}(x_0)\left[\int_{\mathbb{S}}p(x_0,\mathrm{d}x_1)\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_1}(x_1)\right.\\
&\quad\left.\left[\cdots\left[\int_{\mathbb{S}}p(x_{n-2},\mathrm{d}x_{n-1})h(x_{n-1})\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_{n-1}}(x_{n-1})\right]\cdots\right]\right] \\
&\overset{\text{(h)}}{=} \mathbb{E}_\mu\left[\prod_{k=0}^{n-2}\left(\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_k}(X_k)\right)\left(\mathbb{1}_{\{\mathbb{S}\setminus(A\cup B)\}\cap A_{n-1}}(X_{n-1})h(X_{n-1})\right)\right] \\
&\overset{\text{(i)}}{=} \mathbb{E}_\mu\left[h(X_{n-1})\mathbb{1}_{\{V_{A\cup B}\geq n\}}\cdot\mathbb{1}_E\right].
\end{aligned}$$

Each of the steps (e)–(i) can be justified as follows:

(e) the equality (11);

(f) the *equation (5.2.3)* in [1];

(g) from the assumption, we have for every $x_{n-1} \in \{\mathbb{S} \setminus (A \cup B)\} \cap A_{n-1}$,

$$p(x_{n-1}) = \sum_{x_n \in \mathbb{S}} p(x_{n-1}, x_n) h(x_n).$$

(h) the *equation (5.2.3)* in [1];

(i) the equality (11).

Finally, we set $\mathcal{L}_k := \left\{E \in \mathcal{F}_k : \mathbb{E}_\mu\left[h(X_{k+1})\mathbb{1}_{\{V_{A\cup B} \geq k+1\}} \cdot \mathbb{1}_E\right] = \mathbb{E}_\mu\left[h(X_k)\mathbb{1}_{\{V_{A\cup B} \geq k+1\}} \cdot \mathbb{1}_E\right]\right\}$ for each $k \in \mathbb{Z}_+$. Then, the equation (13) yields $\mathcal{P}_{n-1} \subseteq \mathcal{L}_{n-1}$. Now, we claim that $\mathcal{L}_{n-1}$ is a π-system on Ω_0.

1. Since $\mathcal{P}_{n-1} \subseteq \mathcal{L}_{n-1}$, both $\varnothing$ and Ω_0 belong to $\mathcal{L}_{n-1}$;

2. If $E, F \in \mathcal{L}_{n-1}$ with $E \subseteq F$, then we obtain from the linearlity of expectations that

$$\begin{aligned}
\mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_{F\setminus E}\right] &= \mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_F\right] - \mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_E\right] \\
&= \mathbb{E}_\mu\left[h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_F\right] - \mathbb{E}_\mu\left[h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_E\right] \\
&= \mathbb{E}_\mu\left[h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_{F\setminus E}\right],
\end{aligned}$$

   thereby $F \setminus E \in \mathcal{L}_{n-1}$.

3. Let $\{E_k\}_{k=1}^\infty$ be a sequence in $\mathcal{L}_{n-1}$ with $E_k \uparrow E$ as $k \to \infty$. Then, $\mathbb{1}_{E_k} \overset{k\to\infty}{\to} \mathbb{1}_E$, and so the bounded convergence theorem yields

$$\begin{aligned}
\mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_E\right] &= \lim_{k\to\infty} \mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_{E_k}\right] \\
&= \lim_{k\to\infty} \mathbb{E}_\mu\left[h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_{E_k}\right] \\
&= \mathbb{E}_\mu\left[h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n\}} \cdot \mathbb{1}_E\right],
\end{aligned}$$

   thereby $E \in \mathcal{L}_{n-1}$.

Hence, $\mathcal{L}_{n-1}$ is a λ-system on Ω_0 with $\mathcal{P}_{n-1} \subseteq \mathcal{L}_{n-1}$ and so we get $\mathcal{L}_{n-1} = \sigma(\mathcal{P}_{n-1}) = \mathcal{F}_{n-1}$ by the π-λ theorem (*Theorem 2.1.6* in [1]). Since $h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n\}} = h(X_{n-1)}\left(1 - \mathbb{1}_{\{V_{A\cup B} \leq n-1\}}\right)$ is $\mathcal{F}_{n-1}$-measurable, it establishes the desired claim.

□

Putting Claim 2 into the equation (10) yields

$$\begin{aligned}
\mathbb{E}_\mu\left[M_n \middle| \mathcal{F}_{n-1}\right] &= \mathbb{E}_\mu\left[h(X_n)\mathbb{1}_{\{V_{A\cup B} \geq n\}} \middle| \mathcal{F}_{n-1}\right] + \sum_{k=0}^{n-1} h(X_k)\mathbb{1}_{\{V_{A\cup B} = k\}} \\
&= h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n\}} + \sum_{k=0}^{n-1} h(X_k)\mathbb{1}_{\{V_{A\cup B} = k\}} \\
&= h(X_{n-1})\mathbb{1}_{\{V_{A\cup B} \geq n-1\}} + \sum_{k=0}^{n-2} h(X_k)\mathbb{1}_{\{V_{A\cup B} = k\}} \\
&= h\left(X_{(n-1)\wedge V_{A\cup B}}\right) = M_{n-1}
\end{aligned}$$

$\mathbb{P}_\mu$-almost surely. Hence, the stochastic process $\{M_n = h(X_{n\wedge V_{A\cup B}})\}_{n=0}^\infty$ is a martingale with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ for any bounded function $h : \mathbb{S} \to \mathbb{R}$ satisfying the condition $(*)$.

(iii) Let $g : \mathbb{S} \to \mathbb{R}$ be any other function satisfying the condition $(*)$, and $g(x) = 1$ if $x \in A$; $g(x) = 0$ if $x \in B$. Since

$$\sup\{|g(x)| : x \in \mathbb{S}\} \le \max\{1, \sup\{|g(x)| : x \in \mathbb{S} \setminus (A \cup B)\}\} \overset{\text{(j)}}{<} +\infty,$$

where the step (j) holds since $\mathbb{S} \setminus (A \cup B)$ is finite, g is a bounded function and so is h by the same argument. If we let $f := g - h$, then $f : \mathbb{S} \to \mathbb{R}$ is a bounded function satisfying the condition $(*)$ together with $f(x) = 0$ for $x \in A \cup B$. As we have shown that the second statement (ii) of this problem is valid for any bounded function from $\mathbb{S}$ to $\mathbb{R}$ which satisfies the condition $(*)$, $\{f(X_{n\wedge V_{A\cup B}})\}_{n=0}^\infty$ is a martingale with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. Thus for any $x \in \mathbb{S} \setminus (A \cup B)$, we have

$$\begin{aligned} f(x) &= \mathbb{E}_x\left[f\left(X_{0\wedge V_{A\cup B}}\right)\right] \\ &= \mathbb{E}_x\left[f\left(X_{n\wedge V_{A\cup B}}\right)\right] \\ &= \mathbb{E}_x\left[f(X_n)\mathbb{1}_{\{V_{A\cup B} > n\}}\right] + \mathbb{E}_x\left[f\left(X_{V_{A\cup B}}\right)\mathbb{1}_{\{V_{A\cup B} \le n\}}\right] \\ &\overset{\text{(k)}}{=} \mathbb{E}_x\left[f(X_n)\mathbb{1}_{\{V_{A\cup B} > n\}}\right], \end{aligned} \tag{14}$$

where the step (h) follows from the fact that if $V_{A\cup B} < +\infty$, then $X_{V_{A\cup B}} \in A \cup B$ and so $f(X_{V_{A\cup B}}) = 0$. Since f is bounded, $L := \sup\{|f(x)| : x \in \mathbb{S}\} < +\infty$. Then, one has from (14) that

$$\begin{aligned} |f(x)| &= \left|\mathbb{E}_x\left[f(X_n)\mathbb{1}_{\{V_{A\cup B} > n\}}\right]\right| \\ &\le \mathbb{E}_x\left[|f(X_n)|\,\mathbb{1}_{\{V_{A\cup B} > n\}}\right] \\ &\le L \cdot \mathbb{P}_x\{V_{A\cup B} > n\} \\ &\overset{\text{(l)}}{=} L \cdot \mathbb{P}_x\{T_{A\cup B} > n\} \end{aligned} \tag{15}$$

for every $n \in \mathbb{Z}_+$, where the step (l) comes from $x \in \mathbb{S}\setminus(A \cup B)$. As $\mathbb{S}\setminus(A \cup B)$ is finite and $\mathbb{P}_y\{T_{A\cup B} < +\infty\} > 0$ for all $y \in \mathbb{S} \setminus (A \cup B)$, we can apply Problem 3 (*Exercise 5.2.6* in [1]) at this point: there exists an $N \in \mathbb{N}$ and $\epsilon > 0$ such that $\mathbb{P}_y\{T_{A\cup B} > kN\} \le (1-\epsilon)^k$ for all $k \in \mathbb{N}$ and $y \in \mathbb{S} \setminus (A \cup B)$. Putting $n = kN$ into the bound (15) yields for every $x \in \mathbb{S} \setminus (A \cup B)$ that

$$|f(x)| \le L(1-\epsilon)^k \tag{16}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ in (16), we can conclude that $f(x) = 0$ for all $x \in \mathbb{S} \setminus (A \cup B)$. Consequently, we have $f(x) = g(x) - h(x) = 0$ for all $x \in \mathbb{S}$, thereby $g \equiv h$ on $\mathbb{S}$. This completes the proof of the third statement.

Problem 5 (*Exercise 5.2.8.* in [1])**.**

Let $\mathbb{S} := [0 : N] = \{0, 1, \cdots, N-1\}$. Then, one can see that

- $\{0\} \cap \{N\} = \varnothing$;
- $\mathbb{S} \setminus \{0, N\} = \{1, 2, \cdots, N-1\}$ is finite;
- Since $V_0 \wedge V_N = V_{\{0\}\cup\{N\}}$, $\mathbb{P}_x\left\{V_{\{0\}\cup\{N\}} < +\infty\right\} = \mathbb{P}_x\{V_0 \wedge V_N < +\infty\} > 0$ for all $x \in \mathbb{S} \setminus \{0, N\}$.

According to the above observations and the third problem (3) of Problem 4 (*Exercise 5.2.7* in [1]), we know that the function $h : \mathbb{S} \to \mathbb{R}$ defined by $h(x) := \mathbb{P}_x \{V_N < V_0\}$, $x \in \mathbb{S}$, is the unique function such that $h(0) = 0$, $h(N) = 1$, and

$$h(x) = \sum_{y \in \mathbb{S}} p(x, y) h(y), \ \forall x \in \mathbb{S} \setminus \{0, N\}, \tag{17}$$

where $p(\cdot, \cdot) : \mathbb{S} \times \mathbb{S} \to [0, 1]$ denotes the transition probability of given homogeneous Markov chain.

Now, let $g : \mathbb{S} \to \mathbb{R}$ to be $g(x) := \frac{x}{N}$, $x \in \mathbb{S}$. It's clear that $g(0) = 0$ and $g(N) = 1$. We claim that the function $g : \mathbb{S} \to \mathbb{R}$ satisfies the equation (17). Since $\{X_n\}_{n=0}^{\infty}$ is a martingale with respect to the canonical filtration $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^{\infty}$, we have

$$\begin{aligned} X_{n-1} &= \mathbb{E}_\mu [X_n | \mathcal{F}_{n-1}] \\ &= \mathbb{E}_\mu \left[X_n \left(\sum_{y \in \mathbb{S}} \mathbb{1}_{\{X_n = y\}} \right) \middle| \mathcal{F}_{n-1} \right] \\ &\overset{\text{(a)}}{=} \sum_{y \in \mathbb{S}} \mathbb{E}_\mu \left[X_n \mathbb{1}_{\{X_n = y\}} \middle| \mathcal{F}_{n-1} \right] \\ &= \sum_{y \in \mathbb{S}} \mathbb{E}_\mu \left[y \mathbb{1}_{\{X_n = y\}} \middle| \mathcal{F}_{n-1} \right] \\ &= \sum_{y \in \mathbb{S}} y \cdot \mathbb{P}_\mu \{X_n = y | \mathcal{F}_{n-1}\} \\ &\overset{\text{(b)}}{=} \sum_{y \in \mathbb{S}} y \cdot p(X_{n-1}, y), \end{aligned} \tag{18}$$

$\mathbb{P}_\mu$-almost surely, where μ is any initial distribution of the Markov chain $\{X_n\}_{n=0}^{\infty}$, and the step (a) is valid since $\mathbb{S} = [0 : N]$ is finite, the step (b) follows from the assumption that $\{X_n\}_{n=0}^{\infty}$ is a homogeneous Markov chain with transition probability $p(\cdot, \cdot)$. Thus, it follows from (18) that

$$\begin{aligned} x &= \int_{\{X_{n-1} = x\}} X_{n-1} \mathrm{d}\mathbb{P}_\mu \\ &= \int_{\{X_{n-1} = x\}} \left[\sum_{y \in \mathbb{S}} y \cdot p(X_{n-1}, y) \right] \mathrm{d}\mathbb{P}_\mu \\ &= \sum_{y \in \mathbb{S}} y \int_{\{X_{n-1} = x\}} p(X_{n-1}, y) \, \mathrm{d}\mathbb{P}_\mu \\ &= \sum_{y \in \mathbb{S}} y \cdot p(x, y) \end{aligned} \tag{19}$$

for every $x \in \mathbb{S}$, since $\{X_{n-1} = x\} \in \mathcal{F}_{n-1}$. Dividing the equation (19) by N yields

$$g(x) = \sum_{y \in \mathbb{S}} p(x, y) g(y), \ \forall x \in \mathbb{S},$$

thereby the function $g : \mathbb{S} \to \mathbb{R}$ satisfies the equation (17). From the uniqueness of such a function h, one can deduce $g \equiv h$ on $\mathbb{S}$. Hence,

$$\mathbb{P}_x \{V_N < V_0\} = h(x) = g(x) = \frac{x}{N}$$

for all $x \in \mathbb{S} = [0 : N]$.

Problem 6 (*Exercise 5.2.11.* in [1]: Exit times)**.**

(i) Fix any $x \in \mathbb{S} \setminus A$, and consider the following two cases:

(Case #1) $\mathbb{P}_x \{V_A = +\infty\} > 0$: Define $Z : \Omega_0 \to \mathbb{R}$ by $Z(\omega) := \mathbb{1}_{\{V_A = +\infty\}}(\omega)$ for $\omega \in \Omega_0$. Then for every $x \in \mathbb{S} \setminus A$,

$$\begin{aligned}
\mathbb{P}_x \{V_A = +\infty\} &= \mathbb{E}_x [Z] \\
&\overset{\text{(a)}}{=} \mathbb{E}_x [Z \circ \theta_1] \\
&= \mathbb{E}_x [\mathbb{E}_x [Z \circ \theta_1 | \mathcal{F}_1]] \\
&\overset{\text{(b)}}{=} \mathbb{E}_x [\mathbb{E}_{X_1} [Z]] \\
&= \mathbb{E}_x \left[\mathbb{E}_{X_1} [Z] \left(\sum_{y \in \mathbb{S}} \mathbb{1}_{\{X_1 = y\}} \right) \right] \\
&\overset{\text{(c)}}{=} \sum_{y \in \mathbb{S}} \mathbb{E}_x \left[\mathbb{E}_{X_1} [Z] \mathbb{1}_{\{X_1 = y\}} \right] \\
&= \sum_{y \in \mathbb{S}} \mathbb{E}_x \left[\mathbb{E}_y [Z] \mathbb{1}_{\{X_1 = y\}} \right] \\
&= \sum_{y \in \mathbb{S}} \underbrace{\mathbb{P}_x \{X_1 = y\}}_{= \, p(x,y)} \mathbb{E}_y [Z] \\
&= \sum_{y \in \mathbb{S}} p(x,y) \cdot \mathbb{P}_y \{V_A = +\infty\} > 0.
\end{aligned} \tag{20}$$

Here, the above steps (a)–(c) can be justified as follows:

(a) Since $x \in \mathbb{S} \setminus A$, $Z = Z \circ \theta_1$ if $X_0 = x$;

(b) Since Z is a bounded, measurable function defined on the sequence space $(\Omega_0, \mathcal{F}_\infty)$, we can apply the Markov property (*Theorem 5.2.3* in [1]) and the step (b) follows;

(c) the Fubini-Tonelli's theorem, since the summands are non-negative.

The inequality (20) implies $p(x,y) \cdot \mathbb{P}_y \{V_A = +\infty\} > 0$ for some $y \in \mathbb{S}$. As a consequence, we have

$$p(x,y) g(y) = p(x,y) \cdot \mathbb{E}_y [V_A] \geq p(x,y) \cdot \mathbb{E}_y \left[V_A \cdot \mathbb{1}_{\{V_A = +\infty\}} \right] \overset{\text{(d)}}{=} +\infty,$$

where the step (d) holds since $p(x,y) > 0$ and $\mathbb{P}_y \{V_A = +\infty\} > 0$. Hence, we arrive at

$$1 + \sum_{y \in \mathbb{S}} p(x,y) g(y) = +\infty \overset{\text{(e)}}{=} \mathbb{E}_x [V_A] = g(x),$$

where the step (e) follows from the assumption $\mathbb{P}_x \{V_A = +\infty\} > 0$.

(Case #2) $\mathbb{P}_x \{V_A = +\infty\} = 0$: Then, we have from the monotone convergence theorem that

$$g(x) = \mathbb{E}_x [V_A] \overset{\text{(f)}}{=} \mathbb{E}_x \left[V_A \cdot \mathbb{1}_{\{V_A < +\infty\}} \right] = \lim_{n \to \infty} \uparrow \mathbb{E}_x \left[V_A \cdot \mathbb{1}_{\{V_A \leq n\}} \right], \tag{21}$$

where the step (f) is due to the assumption $\mathbb{P}_x \{V_A = +\infty\} = 0$. Define $Y_n : \Omega_0 \to \mathbb{R}$ for $n \in \mathbb{Z}_+$ by

$$Y_n(\omega) := V_A(\omega) \cdot \mathbb{1}_{\{V_A \leq n\}}(\omega), \ \forall \omega \in \Omega_0.$$

As $|Y_n| = \left|V_A \cdot \mathbb{1}_{\{V_A \le n\}}\right| \le n \cdot \mathbb{1}_{\{V_A \le n\}}$ on Ω_0, every Y_n is a bounded, measurable function defined on the sequence space $(\Omega_0, \mathcal{F}_\infty)$. One can see that if $X_0 \in \mathbb{S} \setminus A$,

$$\begin{aligned}
(Y_n \circ \theta_1)(\omega) &= V_A(\theta_1(\omega)) \cdot \mathbb{1}_{\{V_A(\theta_1(\omega)) \le n\}} \\
&\overset{\text{(g)}}{=} (V_A(\omega) - 1)\, \mathbb{1}_{\{V_A(\omega) - 1 \le n\}} \\
&= V_A(\omega) \cdot \mathbb{1}_{\{V_A \le n+1\}}(\omega) - \mathbb{1}_{\{V_A \le n+1\}}(\omega) \\
&= Y_{n+1}(\omega) - \mathbb{1}_{\{V_A \le n+1\}}(\omega)
\end{aligned}$$

for every $\omega \in \Omega_0$ and $n \in \mathbb{Z}_+$, where the step (g) holds since if $V_A(\omega) \ge 1$, then $V_A(\omega) = V_A(\theta_1(\omega)) + 1$. Thus, $Y_n = (Y_{n-1} \circ \theta_1) + \mathbb{1}_{\{V_A \le n\}}$ on Ω_0. Hence,

$$\begin{aligned}
\mathbb{E}_x\left[V_A \cdot \mathbb{1}_{\{V_A \le n\}}\right] &= \mathbb{E}_x[Y_n] \\
&= \mathbb{E}_x[Y_{n-1} \circ \theta_1] + \mathbb{P}_x\{V_A \le n\} \\
&= \mathbb{E}_x[\mathbb{E}_x[Y_{n-1} \circ \theta_1 | \mathcal{F}_1]] + \mathbb{P}_x\{V_A \le n\} \\
&\overset{\text{(h)}}{=} \mathbb{E}_x[\mathbb{E}_{X_1}[Y_{n-1}]] + \mathbb{P}_x\{V_A \le n\},
\end{aligned} \tag{22}$$

where the step (h) is due to the Markov property (*Theorem 5.2.3* in [1]). Here, $\{\mathcal{F}_n := \sigma(X_0, X_1, \cdots, X_n)\}_{n=0}^{\infty}$ refers to the canonical filtration of $\{X_n\}_{n=0}^{\infty}$. By letting $n \to \infty$ in the equation (22), it follows that

$$\begin{aligned}
g(x) &\overset{\text{(i)}}{=} \lim_{n\to\infty} \uparrow \mathbb{E}_x\left[V_A \cdot \mathbb{1}_{\{V_A \le n\}}\right] \\
&= \lim_{n\to\infty} \uparrow \mathbb{E}_x[\mathbb{E}_{X_1}[Y_{n-1}]] + \mathbb{P}_x\{V_A < +\infty\} \\
&\overset{\text{(j)}}{=} \mathbb{E}_x\left[\mathbb{E}_{X_1}\left[V_A \cdot \mathbb{1}_{\{V_A < +\infty\}}\right]\right] + 1 \\
&= \mathbb{E}_x\left[\sum_{y \in \mathbb{S}} \mathbb{E}_{X_1}\left[V_A \cdot \mathbb{1}_{\{V_A < +\infty\}}\right] \mathbb{1}_{\{X_1 = y\}}\right] + 1 \\
&\overset{\text{(k)}}{=} \sum_{y \in \mathbb{S}} \mathbb{E}_x\left[\mathbb{E}_{X_1}\left[V_A \cdot \mathbb{1}_{\{V_A < +\infty\}}\right] \mathbb{1}_{\{X_1 = y\}}\right] + 1 \\
&= \sum_{y \in \mathbb{S}} \mathbb{E}_x\left[\mathbb{E}_y\left[V_A \cdot \mathbb{1}_{\{V_A < +\infty\}}\right] \mathbb{1}_{\{X_1 = y\}}\right] + 1 \\
&= \sum_{y \in \mathbb{S}} p(x,y) \mathbb{E}_y\left[V_A \cdot \mathbb{1}_{\{V_A < +\infty\}}\right] + 1 \\
&\overset{\text{(l)}}{=} \sum_{y \in \mathbb{S}} p(x,y) \underbrace{\mathbb{E}_y[V_A]}_{= \, g(y)} + 1,
\end{aligned}$$

thereby it establishes our desired result. The steps (i)–(l) can be validated via the following reasons:

(i) the equality (21);

(j) the monotone convergence theorem together with the assumption $\mathbb{P}_x\{V_A = +\infty\} = 0$;

(k) the Fubini-Tonelli's theorem, since the summands are non-negative;

(l) to see this step, we should verify that $\mathbb{P}_y\{V_A < +\infty\} = 1$ for all $y \in \mathbb{S}$. By subtracting (20) from

1, we obtain

$$
\begin{aligned}
1 &= \mathbb{P}_x \{V_A < +\infty\} \\
&= 1 - \mathbb{P}_x \{V_A = +\infty\} \\
&= \sum_{y\in\mathbb{S}} p(x,y) - \sum_{y\in\mathbb{S}} p(x,y) \cdot \mathbb{P}_y \{V_A = +\infty\} \\
&= \sum_{y\in\mathbb{S}} p(x,y) \cdot \mathbb{P}_y \{V_A < +\infty\} \\
&\leq \sum_{y\in\mathbb{S}} p(x,y) \\
&= 1,
\end{aligned}
\tag{23}
$$

so all the inequalities in (23) are in fact equalities. Thus, we get $\mathbb{P}_y \{V_A < +\infty\} = 1$ for all $y \in \mathbb{S}$.

Combining all the arguments of the above two cases completes the proof of the problem (i).

(ii) For convenience, we define $M_n := g(X_{n\wedge V_A}) + (n \wedge V_A)$ for $n \in \mathbb{Z}_+$, for any function $g : \mathbb{S} \to [0, +\infty)$ satisfying the given condition $(*)$. Then, M_n can be written by

$$
M_n = \{g(X_n) + n\} \mathbb{1}_{\{V_A > n\}} + \sum_{k=0}^{n} \{g(X_k) + k\} \mathbb{1}_{\{V_A = k\}}. \tag{24}
$$

Let $L := \sup\{|g(x)| : x \in \mathbb{S} \setminus A\}$, which is finite since $\mathbb{S} \setminus A$ is a finite set. If $V_A > n$, then $X_n \in \mathbb{S} \setminus A$ and it follows that

$$
\left|\{g(X_n) + n\} \mathbb{1}_{\{V_A > n\}}\right| \leq (L + n) \cdot \mathbb{1}_{\{V_A > n\}}. \tag{25}
$$

Also, since $\{V_A = k\} = \{X_0 \in \mathbb{S} \setminus A, \cdots, X_{k-1} \in \mathbb{S} \setminus A, X_k \in A\}$, we have

$$\begin{aligned}
\mathbb{E}_\mu\left[g(X_k)\mathbb{1}_{\{V_A=k\}}\right] &= \mathbb{E}_\mu\left[\prod_{j=0}^{k-1}\mathbb{1}_{\mathbb{S}\setminus A}(X_j)\left\{\mathbb{1}_A(X_k)g(X_k)\right\}\right] \\
&\leq \mathbb{E}_\mu\left[\prod_{j=0}^{k-1}\mathbb{1}_{\mathbb{S}\setminus A}(X_j)g(X_k)\right] \\
&\overset{\text{(m)}}{=} \sum_{x_0\in\mathbb{S}\setminus A}\mu(x_0)\left[\sum_{x_1\in\mathbb{S}\setminus A}p(x_0,x_1)\right. \\
&\left.\left[\cdots\left[\sum_{x_{k-1}\in\mathbb{S}\setminus A}p(x_{k-2},x_{k-1})\underbrace{\left[\sum_{x_k\in\mathbb{S}}p(x_{k-1},x_k)g(x_k)\right]}_{=\,g(x_{k-1})}\right]\cdots\right]\right] \\
&\overset{\text{(n)}}{=} \sum_{x_0\in\mathbb{S}\setminus A}\mu(x_0)\left[\sum_{x_1\in\mathbb{S}\setminus A}p(x_0,x_1)\right. \\
&\left.\left[\cdots\left[\sum_{x_{k-1}\in\mathbb{S}\setminus A}p(x_{k-2},x_{k-1})g(x_{k-1})\right]\cdots\right]\right] \\
&\overset{\text{(o)}}{=} \mathbb{E}_\mu\left[\sum_{j=0}^{k-1}\mathbb{1}_{\mathbb{S}\setminus A}(X_j)g(X_{k-1})\right] \\
&= \mathbb{E}_\mu\left[g(X_{k-1})\mathbb{1}_{\{V_A=k\}}\right] \\
&\overset{\text{(p)}}{\leq} L\cdot\mathbb{1}_{\{V_A=k\}}.
\end{aligned} \tag{26}$$

Here, the above steps (m)–(p) are based on the following reasons:

(m) the equation *(5.2.3)* in [1];

(n) the function g obeys the condition $(*)$;

(o) the equation *(5.2.3)* in [1];

(p) if $V_A = k$, then $X_{k-1} \in \mathbb{S} \setminus A$ and so $g(X_{k-1})\mathbb{1}_{\{V_A=k\}} \leq L \cdot \mathbb{1}_{\{V_A=k\}}$.

Combining (24) together with two pieces (25) and (26) yields for every $n \in \mathbb{Z}_+$,

$$|M_n| \leq (L+n)\mathbb{1}_{\{V_A>n\}} + \sum_{k=0}^{n}(L+k)\mathbb{1}_{\{V_A=k\}} \leq L+n,$$

thereby $M_n \in L^1(\Omega_0, \mathcal{F}_\infty, \mathbb{P}_\mu)$, *i.e.*, each M_n is $\mathbb{P}_\mu$-integrable. Now, we will prove that $\mathbb{E}_\mu\left[M_n \middle| \mathcal{F}_{n-1}\right] \overset{\mathbb{P}_\mu\text{-a.s.}}{=} M_{n-1}$ for all $n \in \mathbb{N}$. We begin by noting that

$$\begin{aligned}
\mathbb{E}_\mu\left[M_n \middle| \mathcal{F}_{n-1}\right] &= \mathbb{E}_\mu\left[\{g(X_n)+n\}\mathbb{1}_{\{V_A\geq n\}}\middle|\mathcal{F}_{n-1}\right] + \sum_{k=0}^{n-1}\mathbb{E}_\mu\left[\{g(X_k)+k\}\mathbb{1}_{\{V_A=k\}}\middle|\mathcal{F}_{n-1}\right] \\
&\overset{\text{(q)}}{=} \mathbb{E}_\mu\left[\{g(X_n)+n\}\mathbb{1}_{\{V_A\geq n\}}\middle|\mathcal{F}_{n-1}\right] + \sum_{k=0}^{n-1}\{g(X_k)+k\}\mathbb{1}_{\{V_A=k\}}
\end{aligned} \tag{27}$$

$\mathbb{P}_\mu$-almost surely, where the step (q) follows from the fact that $\{g(X_k)+k\}\,\mathbb{1}_{\{V_A=k\}}$ is $\mathcal{F}_k$-measurable for $k\in[0:n-1]$. At this point, we claim the following statement.

Claim 3. $\mathbb{E}_\mu\left[\{g(X_n)+n\}\,\mathbb{1}_{\{V_A\geq n\}}\middle|\,\mathcal{F}_{n-1}\right]\overset{\mathbb{P}_\mu\text{-a.s.}}{=}\{g(X_{n-1})+(n-1)\}\,\mathbb{1}_{\{V_A\geq n\}}$.

Proof of Claim 3.

To begin with, we notice that $\{g(X_{n-1})+(n-1)\}\,\mathbb{1}_{\{V_A\geq n\}}$ is $\mathcal{F}_{n-1}$-measurable. As in the proof of Claim 2, let $\mathcal{P}_k:=\left\{\{\omega\in\Omega_0:\omega_0\in A_0,\omega_1\in A_1,\cdots,\omega_k\in A_k\}:A_0,A_1,\cdots,A_k\in\mathcal{S}=2^{\mathbb{S}}\right\}$ for $k\in\mathbb{Z}_+$. Then, $\mathcal{P}_k$ is a π-system on Ω_0 with $\mathcal{F}_k=\sigma(\mathcal{P}_k)$. Also, define

$$\mathcal{L}_k:=\left\{E\in\mathcal{F}_k:\mathbb{E}_\mu\left[\{g(X_{k+1})+(k+1)\}\,\mathbb{1}_{\{V_A\geq k+1\}}\cdot\mathbb{1}_E\right]=\mathbb{E}_\mu\left[\{g(X_k)+k\}\,\mathbb{1}_{\{V_A\geq k+1\}}\cdot\mathbb{1}_E\right]\right\},\ \forall k\in\mathbb{Z}_+.$$

It suffices to show that $\mathcal{L}_{n-1}=\mathcal{F}_{n-1}$. As a next step, we prove $\mathcal{P}_{n-1}\subseteq\mathcal{L}_{n-1}$. Given any $E\in\mathcal{P}_{n-1}$, it can be written by

$$E=\{\omega\in\Omega_0:\omega_0\in A_0,\omega_1\in A_1,\cdots,\omega_{n-1}\in A_{n-1}\}$$

for some $A_0,A_1,\cdots,A_{n-1}\in\mathcal{S}=2^{\mathbb{S}}$. It's clear from $\{V_A\geq n\}=\{X_0\in\mathbb{S}\setminus A,X_1\in\mathbb{S}\setminus A,\cdots,X_{n-1}\in\mathbb{S}\setminus A\}$ that

$$\mathbb{1}_{\{V_A\geq n\}}(\omega)=\sum_{k=0}^{n-1}\mathbb{1}_{\mathbb{S}\setminus A}(\omega_k),\ \forall\omega\in\Omega_0.$$

Hence, we arrive at

$$\begin{aligned}
\mathbb{E}_\mu\left[\{g(X_n)+n\}\,\mathbb{1}_{\{V_A\geq n\}}\cdot\mathbb{1}_E\right]&=\mathbb{E}_\mu\left[\left(\prod_{k=0}^{n-1}\mathbb{1}_{(\mathbb{S}\setminus A)\cap A_k}(X_k)\right)\{g(X_n)+n\}\right]\\
&\overset{\text{(r)}}{=}\sum_{x_0\in(\mathbb{S}\setminus A)\cap A_0}\mu(x_0)\left[\sum_{x_1\in(\mathbb{S}\setminus A)\cap A_1}p(x_0,x_1)\right.\\
&\left.\left[\cdots\left[\sum_{x_{n-1}\in(\mathbb{S}\setminus A)\cap A_{n-1}}p(x_{n-2},x_{n-1})\left[\sum_{x_n\in\mathbb{S}}p(x_{n-1},x_n)\{g(x_n)+n\}\right]\right]\cdots\right]\right]\\
&\overset{\text{(s)}}{=}\sum_{x_0\in(\mathbb{S}\setminus A)\cap A_0}\mu(x_0)\left[\sum_{x_1\in(\mathbb{S}\setminus A)\cap A_1}p(x_0,x_1)\right.\\
&\left.\left[\cdots\left[\sum_{x_{n-1}\in(\mathbb{S}\setminus A)\cap A_{n-1}}p(x_{n-2},x_{n-1})\{g(x_{n-1})+(n-1)\}\right]\cdots\right]\right]\\
&\overset{\text{(t)}}{=}\mathbb{E}_\mu\left[\left(\prod_{k=0}^{n-1}\mathbb{1}_{(\mathbb{S}\setminus A)\cap A_k}(X_k)\right)\{g(X_{n-1})+(n-1)\}\right]\\
&=\mathbb{E}_\mu\left[\{g(X_{n-1})+(n-1)\}\,\mathbb{1}_{\{V_A\geq n\}}\cdot\mathbb{1}_E\right],
\end{aligned}$$

thereby $E\in\mathcal{L}_{n-1}$. Each steps (r)–(t) are valid since:

(r) the equation *(5.2.3)* in [1];

(s) for $x_{n-1}\in\mathbb{S}\setminus A$, we have

$$\begin{aligned}
\sum_{x_n\in\mathbb{S}}p(x_{n-1},x_n)\{g(x_n)+n\}&=\left[1+\sum_{x_n\in\mathbb{S}}p(x_{n-1},x_n)\right]+(n-1)\\
&=g(x_{n-1})+(n-1),
\end{aligned}$$

because the function g satisfies the condition $(*)$;

(t) the equation *(5.2.3)* in [1].

Therefore, $\mathcal{P}_{n-1} \subseteq \mathcal{L}_{n-1}$. From the same argument as in the proof of Claim 2, one can easily see that $\mathcal{L}_{n-1}$ is a λ-system on Ω_0. Employing the π-λ theorem, we eventually obtain $\mathcal{L}_{n-1} = \sigma(\mathcal{P}_{n-1}) = \mathcal{F}_{n-1}$, and this completes the proof of Claim 3.

□

Finally, we can finish the proof of the statement (ii) of this problem. Indeed, from (27) one has

$$\begin{aligned}
\mathbb{E}_\mu \left[M_n \middle| \mathcal{F}_{n-1} \right] &= \mathbb{E}_\mu \left[\{g(X_n) + n\} \mathbb{1}_{\{V_A \geq n\}} \middle| \mathcal{F}_{n-1} \right] + \sum_{k=0}^{n-1} \{g(X_k) + k\} \mathbb{1}_{\{V_A = k\}} \\
&\overset{\text{(u)}}{=} \{g(X_{n-1}) + (n-1)\} \mathbb{1}_{\{V_A \geq n\}} + \sum_{k=0}^{n-1} \{g(X_k) + k\} \mathbb{1}_{\{V_A = k\}} \\
&= \{g(X_{n-1}) + (n-1)\} \mathbb{1}_{\{V_A \geq n-1\}} + \sum_{k=0}^{n-2} \{g(X_k) + k\} \mathbb{1}_{\{V_A = k\}} \\
&= M_{n-1}
\end{aligned}$$

$\mathbb{P}_\mu$-almost surely, where the step (u) follows from Claim 3. So, $\{M_n = g\left(X_{n \wedge V_A}\right) + (n \wedge V_A)\}_{n=0}^{\infty}$ is a martingale with respect to the canonical filtration $\{\mathcal{F}_n = \sigma\left(X_0, X_1, \cdots, X_n\right)\}_{n=0}^{\infty}$ for any function $g : \mathbb{S} \to [0, +\infty)$ that satisfies the condition $(*)$. As a final remark, we note that the statement (ii) also holds for any bounded function $g : \mathbb{S} \to \mathbb{R}$ which satisfies the condition $(*)$. The $\mathbb{P}_\mu$-integrability is immediate from the boundedness of g, and the remaining steps are completely identical. Hence, $\{g\left(X_{n \wedge V_A}\right) + (n \wedge V_A)\}_{n=0}^{\infty}$ is a martingale with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$ for any non-negative or bounded function $g : \mathbb{S} \to \mathbb{R}$ obeying the condition $(*)$.

(iii) Let $h : \mathbb{S} \to \mathbb{R}$ be any function satisfying

$$h(x) = 1 + \sum_{y \in \mathbb{S}} p(x, y) h(y), \ \forall x \in \mathbb{S} \setminus A,$$

together with $h(x) = 0$ for all $x \in A$. So, $\sup_{x \in \mathbb{S}} |h(x)| = \sup_{x \in \mathbb{S} \setminus A} |h(x)| < +\infty$, since $\mathbb{S} \setminus A$ is finite. Thus, h is bounded and likewise, g is also a bounded function which satisfies the condition $(*)$ and $g(x) = 0$ for all $x \in A$. As the second statement (ii) of this problem holds for any bounded function satisfying the condition $(*)$, both $\{g\left(X_{n \wedge V_A}\right) + (n \wedge V_A)\}_{n=0}^{\infty}$ and $\{h\left(X_{n \wedge V_A}\right) + (n \wedge V_A)\}_{n=0}^{\infty}$ are martingales with respect to the canonical filtration $\{\mathcal{F}_n = \sigma\left(X_0, X_1, \cdots, X_n\right)\}_{n=0}^{\infty}$. Now, define $f := g - h : \mathbb{S} \to \mathbb{R}$. Then, $\{f\left(X_{n \wedge V_A}\right)\}_{n=0}^{\infty}$ also forms a martingale with respect to the canonical filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$, since

$$f\left(X_{n \wedge V_A}\right) = \{g\left(X_{n \wedge V_A}\right) + (n \wedge V_A)\} - \{h\left(X_{n \wedge V_A}\right) + (n \wedge V_A)\}.$$

Hence for any $x \in \mathbb{S} \setminus A$, we have

$$\begin{aligned} f(x) &= \mathbb{E}_x \left[f\left(X_{0 \wedge V_A}\right) \right] \\ &\overset{\text{(v)}}{=} \mathbb{E}_x \left[f\left(X_{n \wedge V_A}\right) \right] \\ &= \mathbb{E}_x \left[f(X_n) \mathbb{1}_{\{V_A > n\}} \right] + \sum_{k=0}^{n} \mathbb{E}_x \left[f(X_k) \mathbb{1}_{\{V_A = k\}} \right] \\ &\overset{\text{(w)}}{=} \mathbb{E}_x \left[f(X_n) \mathbb{1}_{\{V_A > n\}} \right], \end{aligned} \tag{28}$$

for every $n \in \mathbb{Z}_+$, where the step (v) holds since $\left\{ f\left(X_{n \wedge V_A}\right) \right\}_{n=0}^{\infty}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n=0}^{\infty}$, and the step (w) is owing to the fact that if $V_A = k$, then $X_k \in A$ together with the fact $f(x) = g(x) - h(x) = 0$ for all $x \in A$. As both g and h are bounded, so is f. Thus, $L := \sup \{|f(x)| : x \in \mathbb{S}\} < +\infty$. Then, we obtain from (28) that

$$|f(x)| = \left| \mathbb{E}_x \left[f(X_n) \mathbb{1}_{\{V_A > n\}} \right] \right| \leq \mathbb{E}_x \left[|f(X_n)| \, \mathbb{1}_{\{V_A > n\}} \right] \leq L \cdot \mathbb{P}_x \{V_A > n\} \tag{29}$$

for all $x \in \mathbb{S} \setminus A$ and $n \in \mathbb{Z}_+$.

On the other hand, it's clear that $\mathbb{S} \setminus A$ is finite, and $\mathbb{P}_x \{T_A < +\infty\} = \mathbb{P}_x \{V_A < +\infty\} < +\infty$ for every $x \in \mathbb{S} \setminus A$ from the assumptions of the problem. So, we can apply Problem (3) (*Exercise 5.2.6.* in [1]): there is an $N \in \mathbb{N}$ and $\epsilon > 0$ such that $\mathbb{P}_y \{T_A > kN\} \leq (1 - \epsilon)^k$ for all $k \in \mathbb{N}$ and $y \in \mathbb{S} \setminus C$. Plugging $n = kN$ into the bound (29) produces for every $x \in \mathbb{S} \setminus A$,

$$|f(x)| = L \cdot \mathbb{P}_x \{V_A > kN\} = L \cdot \mathbb{P}_x \{T_A > kN\} \leq L(1 - \epsilon)^k$$

for all $k \in \mathbb{N}$. By letting $k \to \infty$, it gives $f(x) = 0$ for all $x \in \mathbb{S} \setminus A$. Hence, $f = g - h \equiv 0$ on $\mathbb{S}$, thereby $h(x) = g(x) = \mathbb{E}_x [V_A]$ for all $x \in \mathbb{S}$. This establishes our desired result.

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.