# MAS651 Theory of Stochastic Processes
# Homework #11

20150597 Jeonghwan Lee

Department of Mathematical Sciences, KAIST

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set $A$ and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$.

**Problem 1** (*Exercise 8.4.1.* in [1]).

We recall from the definition of Brownian bridges that

$$\mathbb{P}\left\{B^0(\cdot) \in A\right\} = \mathbb{P}\left\{\pi_1\left\{B(\cdot)\right\} \in A \middle| B(1) = 0\right\} \tag{1}$$

for every $A \in \mathcal{C}([0,1], \mathbb{R})$, where $\pi_1(\cdot) : (C([0,+\infty), \mathbb{R}), \mathcal{C}([0,+\infty), \mathbb{R})) \to (C([0,1], \mathbb{R}), \mathcal{C}([0,1], \mathbb{R}))$ refers to the canonical projection map defined by $\pi_1(\Phi) := \Phi|_{[0,1]}$, $\left\{B^0(t) : t \in [0,1]\right\}$ is a Brownian bridge defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{B(t) : t \in \mathbb{R}_+\}$ is a standard one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{M(t) : t \in \mathbb{R}_+\}$ be the maximum process of the standard one-dimensional Brownian motion $\{B(t) : t \in \mathbb{R}_+\}$, *i.e.*,

$$M(t) := \max\left\{B(s) : s \in [0,t]\right\}, \ \forall t \in \mathbb{R}_+.$$

Then, we have

$$\begin{aligned}
\mathbb{P}\left\{\max\left\{B^0(t) : t \in [0,1]\right\} > b\right\} &\overset{\text{(a)}}{=} \mathbb{P}\left\{\max\left\{B(t) : t \in [0,1]\right\} > b \middle| B(1) = 0\right\} \\
&= \mathbb{P}\left\{M(1) > b \middle| B(1) = 0\right\} \\
&= \frac{\mathbb{P}\left\{M(1) > b, B(1) = 0\right\}}{p_1(0,0)} \\
&\overset{\text{(b)}}{=} \frac{\mathbb{P}\left\{T_b < 1, B(1) = 0\right\}}{p_1(0,0)} \\
&\overset{\text{(c)}}{=} \frac{p_1(0,2b)}{p_1(0,0)} \\
&= \frac{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(2b)^2\right\}}{\frac{1}{\sqrt{2\pi}}} \\
&= \exp\left(-2b^2\right),
\end{aligned}$$

as desired, where the steps (a)–(c) can be justified as follows:

(a) the equation (1);

(b) The $\mathbb{P}$-a.s. continuity of the Brownian paths implies $\{M(1) > b\} \cap \mathcal{B} = \{T_b < 1\} \cap \mathcal{B}$, where

$$\mathcal{B} := \{\omega \in \Omega : \text{ the sample path of } \{B(t) : t \in \mathbb{R}_+\} \text{ is continuous everywhere.}\} \in \mathcal{F},$$

since $B(0) = 0$. Thus we have $\mathbb{P}\{\{M(1) > b\} \triangle \{T_b < 1\}\} = 0$;

(c) the *Exercise 7.4.3-(b)* in [1].

**Problem 2** (*Exercise 8.4.2.* in [1]).

Let $\{B(t) : t \in \mathbb{R}_+\}$ be a standard one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{X(t) : t \in [0, 1]\}$ be a continuous-time stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$X(t) := (1-t) B\left(\frac{t}{1-t}\right), \ \forall t \in [0, 1),$$

and $X(1) := 0$. Let $\mathcal{B} := \{\omega \in \Omega : \text{ the sample path of } \{B(t) : t \in \mathbb{R}_+\} \text{ is continuous everywhere}\} \in \mathcal{F}$. It's clear that the sample path of $\{X(t) : t \in [0, 1]\}$ at $\omega \in \mathcal{B}$ is continuous on $[0, 1)$. According to *Theorem 7.2.6* in [1], we know that

$$\lim_{t \downarrow 0^+} t \cdot B\left(\frac{1}{t}\right) = 0 \quad \mathbb{P}\text{-almost surely.}$$

Let $\mathcal{E} := \left\{\omega \in \Omega : \lim_{t \downarrow 0^+} t \cdot B_{\frac{1}{t}}(\omega) = 0\right\} \in \mathcal{F}$. Since $\mathbb{P}\{\mathcal{B}\} = \mathbb{P}\{\mathcal{E}\} = 1$, we obtain $\mathbb{P}\{\mathcal{B} \cap \mathcal{E}\} = 1$ and when $\omega \in \mathcal{B} \cap \mathcal{E}$, we see that

$$\begin{aligned} \lim_{t \uparrow 1^-} X_t(\omega) &= \lim_{t \uparrow 1^-} (1-t) B_{\frac{t}{1-t}}(\omega) \\ &\overset{\text{(a)}}{=} \lim_{u \to +\infty} \frac{B_u(\omega)}{u} \cdot \frac{u}{1+u} \\ &\overset{\text{(b)}}{=} \lim_{v \downarrow 0^+} v \cdot B_{\frac{1}{v}}(\omega) \cdot \frac{1}{1+v} \\ &= 0 = X_1(\omega), \end{aligned} \tag{2}$$

where the step (a) makes use of the substitution $u = \frac{t}{1-t}$, and the step (b) utilizes the substitution $v = \frac{1}{u}$. Thus we may conclude that the sample path of $\{X(t) : t \in [0, 1]\}$ at $\omega \in \mathcal{B} \cap \mathcal{F}$ is continuous everywhere, thereby the sample path of $\{X(t) : t \in [0, 1]\}$ is continuous everywhere for $\mathbb{P}$-a.s. $\omega \in \Omega$.

Now, it remains to show that $\{X(t) : t \in [0, 1]\}$ has the same joint law as $\left\{B^0(t) : t \in [0, 1]\right\}$ under $\mathbb{P}$, where $\left\{B^0(t) : t \in [0, 1]\right\}$ is a Brownian bridge defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Since $X(0) = X(1) = B^0(0) = B^0(1) = 0$ $\mathbb{P}$-almost surely, it suffices to prove that for every $0 < t_1 < t_2 < \cdots < t_n < 1$,

$$(X(t_1), X(t_2), \cdots, X(t_n)) \overset{d}{=} \left(B^0(t_1), B^0(t_2), \cdots, B^0(t_n)\right) \tag{3}$$

under $\mathbb{P}$. Since we know from the *equation 8.4.3* in [1] that $\left\{B^0(t) : t \in [0, 1]\right\}$ and $\{B(t) - t \cdot B(1) : t \in [0, 1]\}$ have the same joint law under $\mathbb{P}$. From

$$\begin{bmatrix} B^0(t_1) \\ B^0(t_2) \\ \vdots \\ B^0(t_n) \end{bmatrix} \overset{d}{=} \begin{bmatrix} B(t_1) - t_1 \cdot B(1) \\ B(t_2) - t_2 \cdot B(1) \\ \vdots \\ B(t_n) - t_n \cdot B(1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & -t_1 \\ 0 & 1 & \cdots & 0 & -t_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -t_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}}_{=: \mathbf{A} \in \mathbb{R}^{n \times (n+1)}} \begin{bmatrix} B(t_1) \\ B(t_2) - B(t_1) \\ \vdots \\ B(t_n) - B(t_{n-1}) \\ B(1) - B(t_n) \end{bmatrix}$$

together with the fact that

$$\begin{bmatrix} B(t_1) \\ B(t_2)-B(t_1) \\ \vdots \\ B(t_n)-B(t_{n-1}) \\ B(1)-B(t_n) \end{bmatrix} \sim \bigotimes_{k=1}^{n+1} \mathsf{N}_1\left(0, t_k - t_{k-1}\right) \stackrel{d}{=} \mathsf{N}_{n+1}\left(\mathbf{0}, \operatorname{diag}\left(t_1, t_2 - t_1, \cdots, t_n - t_{n-1}, 1 - t_n\right)\right),$$

where $t_0 := 0$, $t_1 := 1$, $\mathbf{0} \in \mathbb{R}^{n+1}$ denotes the $(n+1)$-dimensional zero vector, and $\mathsf{N}_d\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$ refers to the $d$-dimensional normal distribution with mean vector $\boldsymbol{\mu} \in \mathbb{R}^d$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}$, we arrive at

$$\begin{bmatrix} B^0(t_1) \\ B^0(t_2) \\ \vdots \\ B^0(t_n) \end{bmatrix} \sim \mathsf{N}_n\left(\mathbf{0}, \mathbf{A} \cdot \operatorname{diag}\left(t_1, t_2 - t_1, \cdots, t_n - t_{n-1}, 1 - t_n\right) \cdot \mathbf{A}^\top\right),$$

where $\mathbf{0} \in \mathbb{R}^n$ denotes the $n$-dimensional zero vector. By involving some straightforward calculations, we obtain

$$\boldsymbol{\Sigma}_1 := \mathbf{A} \cdot \operatorname{diag}\left(t_1, t_2 - t_1, \cdots, t_n - t_{n-1}, 1 - t_n\right) \cdot \mathbf{A}^\top = \left[t_{\min\{i,j\}}\left(1 - t_{\max\{i,j\}}\right)\right]_{(i,j)\in[n]\times[n]}. \tag{4}$$

Note that the equation (4) can be obtained directly from the *equation 8.4.5* in [1].

On the other hand, we can easily observe that $(X(t_1), X(t_2), \cdots, X(t_n))$ follows an $n$-dimensional normal distribution with zero mean vector since

$$\begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_n) \end{bmatrix} = \operatorname{diag}\left(1 - t_1, 1 - t_2, \cdots, 1 - t_n\right) \underbrace{\begin{bmatrix} B\left(\frac{t_1}{1-t_1}\right) \\ B\left(\frac{t_2}{1-t_2}\right) \\ \vdots \\ B\left(\frac{t_n}{1-t_n}\right) \end{bmatrix}}_{\text{normally distributed!}}.$$

So it suffices to compute the covariance matrix of $(X(t_1), X(t_2), \cdots, X(t_n))$, say $\boldsymbol{\Sigma}_2 \in \mathbb{R}^{n\times n}$:

$$\begin{aligned} \left[\boldsymbol{\Sigma}_2\right]_{ij} &= \mathsf{Cov}\left[X(t_i), X(t_j)\right] \\ &= \mathbb{E}\left[(1 - t_i)(1 - t_j) \cdot B\left(\frac{t_i}{1 - t_i}\right) B\left(\frac{t_j}{1 - t_j}\right)\right] \\ &\stackrel{\text{(c)}}{=} (1 - t_i)(1 - t_j) \cdot \min\left\{\frac{t_i}{1 - t_i}, \frac{t_j}{1 - t_j}\right\} \\ &= t_{\min\{i,j\}}\left(1 - t_{\max\{i,j\}}\right) \\ &= \left[\boldsymbol{\Sigma}_1\right]_{ij} \end{aligned}$$

for every $(i,j) \in [n] \times [n]$, where the step (c) follows from the *Property (b')* in pp.355 of [1], thereby we have $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$. This establishes our desired result (3) and this proves the claim that $\{X(t) : t \in [0,1]\}$ has the same joint law as $\left\{B^0(t) : t \in [0,1]\right\}$ under $\mathbb{P}$. Hence, $\{X(t) : t \in [0,1]\}$ is a Brownian bridge defined on $(\Omega, \mathcal{F}, \mathbb{P})$ as desired.

## References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.