# MAS651 Theory of Stochastic Processes
# Homework #10

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, $\mathbb{R}_+$ be the set of all non-negative real numbers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set $A$ and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$.

**Problem 1** (*Exercise 8.1.1.* in [1])**.**

Given any $-\infty < a < 0 \leq b < +\infty$, let $T_{a,b} := \inf\{t \in \mathbb{R}_+ : B(t) \in \mathbb{R} \setminus (a,b)\}$, where $\{B(t) : t \in \mathbb{R}_+\}$ is a standard one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From *Exercise 7.5.4* in [1], we know that

$$\mathbb{E}\left[T_{a,b}^2\right] \leq 4 \cdot \mathbb{E}\left[B\left(T_{a,b}\right)^4\right]. \tag{1}$$

We know that *Exercise 7.5.4* in [1] was one of the problems in *Homework #9* of this course. Let $(U, V)$ be an $((-\infty, 0) \times [0, +\infty), \mathcal{B}((-\infty, 0) \times [0, +\infty)))$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, which is independent of $\{B(t) : t \in \mathbb{R}_+\}$, and has the following distribution:

$$\mathbb{P}\{(U,V) \in A\} = \frac{1}{c} \iint_A (v-u) \mathrm{d}F(u) \mathrm{d}F(v),\ \forall A \in \mathcal{B}((-\infty, 0) \times [0, +\infty)), \tag{2}$$

where $F(\cdot) : \mathbb{R} \to [0,1]$ is the probability distribution function of the random variable $X$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[X] = 0$ and $\mathbb{E}\left[X^2\right] < +\infty$, and $c \in [0, +\infty)$ is the normalization constant given by

$$c := \int_{[0,+\infty)} v \mathrm{d}F(v) \overset{\text{(a)}}{=} -\int_{(-\infty,0)} u \mathrm{d}F(u),$$

where the step (a) follows from the condition $\mathbb{E}[X] = 0$. At this point, we may assume that $\mathbb{E}\left[X^2\right] > 0$ as otherwise trivial. For this case, we know that $c > 0$ and one can check that the function

$$A \in \mathcal{B}((-\infty, 0) \times [0, +\infty)) \mapsto \frac{1}{c} \iint_A (v-u) \mathrm{d}F(u) \mathrm{d}F(v) \in \mathbb{R}_+$$

is a probability measure defined on $((-\infty, 0) \times [0, +\infty), \mathcal{B}((-\infty, 0) \times [0, +\infty)))$. We note that this procedure guarantees the existence of a random variable $(U,V)$ satisfying (2). To this end,

$$\begin{aligned}\iint_{(-\infty,0)\times[0,+\infty)}(v-u)\mathrm{d}F(u)\mathrm{d}F(v) &= \int_{[0,+\infty)}\left[v\cdot\mathbb{P}\{X<0\}-\underbrace{\int_{(-\infty,0)}u\mathrm{d}F(u)}_{=c}\right]\mathrm{d}F(v)\\ &= \mathbb{P}\{X<0\}\underbrace{\int_{[0,+\infty)}v\mathrm{d}F(v)}_{=c}+c\cdot\mathbb{P}\{X\geq 0\}\\ &= c\cdot\mathbb{P}\{X<0\}+c\cdot\mathbb{P}\{X\geq 0\}\\ &= c,\end{aligned}$$

and this establishes our claim. We have shown in the proof of *Theorem 8.1.1* in [1] that for every bounded measurable function $\varphi(\cdot):(\mathbb{R},\mathcal{B}(\mathbb{R}))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$,

$$\int_{\mathbb{R}}\varphi(x)\mathrm{d}F(x)=\mathbb{E}\left[\varphi(X)\right]=\mathbb{E}_{(U,V)}\left[\int_{\{U,V\}}\varphi(x)\mu_{U,V}\left(\mathrm{d}x\right)\right], \tag{3}$$

where the probability measure $\mu_{u,v}$ on $\left(\{u,v\},2^{\{u,v\}}\right)$, $-\infty<u<0\leq v<+\infty$, is defined by

$$\mu_{u,v}\left(\{u\}\right)=\frac{v}{v-u}\quad\text{and}\quad\mu_{u,v}\left(\{v\}\right)=\frac{-u}{v-u}.$$

Note that for any $-\infty<u<0\leq v<+\infty$, we see that

$$\begin{aligned}\mathbb{P}\left\{B\left(T_{u,v}\right)=u\right\}&=\mathbb{P}\left\{T_u<T_v\right\}\overset{\text{(b)}}{=}\frac{v}{v-u}=\mu_{u,v}\left(\{u\}\right);\\ \mathbb{P}\left\{B\left(T_{u,v}\right)=v\right\}&=\mathbb{P}\left\{T_u>T_v\right\}\overset{\text{(c)}}{=}\frac{-u}{v-u}=\mu_{u,v}\left(\{v\}\right),\end{aligned}$$

where the step (b) and the step (c) holds by *Theorem 7.5.3* in [1] (we note that the case $-\infty<u<0<v<+\infty$ makes use of *Theorem 7.5.3* in [1], and the case $-\infty<u<0=v<+\infty$ is trivial). Therefore, $\mu_{u,v}(\cdot)$ is the probability distribution of $B\left(T_{u,v}\right)$ under $\mathbb{P}$ for every $-\infty<u<0\leq v<+\infty$. Thus for every bounded measurable function $\varphi(\cdot):(\mathbb{R},\mathcal{B}(\mathbb{R}))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$,

$$\begin{aligned}\mathbb{E}\left[\varphi(X)\right]&=\mathbb{E}\left[\mathbb{E}\left[\int_{\{U,V\}}\varphi(x)\mu_{U,V}\left(\mathrm{d}x\right)\middle|(U,V)\right]\right]\\ &=\mathbb{E}\left[\mathbb{E}\left[\varphi\left\{B\left(T_{U,V}\right)\right\}\middle|(U,V)\right]\right]\\ &=\mathbb{E}\left[\varphi\left\{B\left(T_{U,V}\right)\right\}\right],\end{aligned}$$

thereby $X\overset{d}{=}B\left(T_{U,V}\right)$ under $\mathbb{P}$. Hence,

$$\begin{aligned}\mathbb{E}\left[T_{U,V}^2\right]&=\mathbb{E}\left[\mathbb{E}\left[T_{U,V}^2\middle|(U,V)\right]\right]\\ &\overset{\text{(d)}}{\leq}\mathbb{E}\left[4\cdot\mathbb{E}\left[B\left(T_{U,V}\right)^4\middle|(U,V)\right]\right]\\ &=4\cdot\mathbb{E}\left[B\left(T_{U,V}\right)^4\right]\\ &\overset{\text{(e)}}{\leq}4\cdot\mathbb{E}\left[X^4\right],\end{aligned}$$

where the step (d) follows from the inequality (1), and the step (e) is due to the fact $X \overset{d}{=} B(T_{U,V})$ under $\mathbb{P}$. We finally note that $T_{U,V}$ is a stopping time with respect to the natural filtration $\{\mathcal{F}(t) : t \in \mathbb{R}_+\}$ generated by $\{B(t) : t \in \mathbb{R}_+\}$, conditionally given $(U, V) = (u, v) \in (-\infty, 0) \times [0, +\infty)$.

**Problem 2** (*Exercise 8.1.2.* in [1]).

Let $X$ be a random variable defined on $(\Omega, \mathcal{F})$ with $\mathbb{E}[X] = 0$ and $\mathbb{E}\left[X^2\right] = \sigma^2 < +\infty$, and $\{X_n\}_{n=1}^{\infty} \overset{\text{i.i.d.}}{\sim} X$ under $\mathbb{P}$. Let $S_0 := 0$ and $S_n := \sum_{k=1}^{n}$ for $n \in \mathbb{N}$. From this one-dimensional simple random walk $\{S_n\}_{n=0}^{\infty}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the following continuous-time stochastic process $\{S(t) : t \in \mathbb{R}_+\}$ on $(\Omega, \mathcal{F})$ by

$$S(t) := \{1 - (t - \lfloor t \rfloor)\} S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) S_{\lfloor t \rfloor + 1}, \ \forall t \in \mathbb{R}_+.$$

Given any $n \in \mathbb{N}$, let $\left\{\hat{S}_n(t) : t \in \mathbb{R}_+\right\}$ be a continuous-time stochastic process $\{S(t) : t \in \mathbb{R}_+\}$ defined on $(\Omega, \mathcal{F})$ by

$$\hat{S}_n(t) := \frac{S(nt)}{\sqrt{n}}, \ \forall t \in \mathbb{R}_+.$$

The Donsker's invariance principle (*Theorem 8.1.4* in [1]) implies

$$\pi_1 \left\{\hat{S}_n(\cdot)\right\} \xrightarrow{d} \sigma \cdot \pi_1 \{B(\cdot)\} \quad \text{as } n \to \infty,$$

where $\pi_L : (C([0, +\infty), \mathbb{R}), \mathcal{C}([0, +\infty), \mathbb{R})) \to (C([0, L], \mathbb{R}), \mathcal{C}([0, L], \mathbb{R}))$ is the *cutting-off map* defined by $\pi_L(f(\cdot)) := f|_{[0,L]}$ for each $L \in \mathbb{R}_+$, and $B(\cdot) : (\Omega, \mathcal{F}) \to (C([0, +\infty), \mathbb{R}), \mathcal{C}([0, +\infty), \mathbb{R}))$ is a map defined by

$$B(\cdot)(\omega)(t) := \begin{cases} B_t(\omega) & \text{if } \omega \in \mathcal{B}; \\ 0 & \text{otherwise,} \end{cases}$$

where $\{B(t) : t \in \mathbb{R}_+\}$ is a standard one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and

$$\mathcal{B} := \{\omega \in \Omega : \text{ the sample path of } \{B(t) : t \in \mathbb{R}_+\} \text{ is continuous everywhere.}\} \in \mathcal{F}.$$

Now, it's time to solve the given problem. One can see that

$$\begin{aligned} R_n &:= 1 + \max\{S_k : k \in [0 : n]\} - \min\{S_k : k \in [0 : n]\} \\ &\overset{\text{(a)}}{=} 1 + \max\{S(t) : t \in [0, n]\} - \min\{S(t) : t \in [0, n]\} \\ &= 1 + \max\{S(nt) : t \in [0, 1]\} - \min\{S(nt) : t \in [0, 1]\}, \end{aligned}$$

where the step (a) follows from the fact that $\{S(t) : t \in [0, n]\}$ attains its maximum and minimum at some integer values of $t \in [0, n]$, since the continuous-time stochastic process $\{S(t) : t \in \mathbb{R}_+\}$ is defined by interpolating the given random walk $\{S_n\}_{n=0}^{\infty}$ linearly. Thus, we have

$$\begin{aligned} \frac{R_n}{\sigma\sqrt{n}} &= \frac{1}{\sigma\sqrt{n}} + \max\left\{\frac{S(nt)}{\sigma\sqrt{n}} : t \in [0, 1]\right\} - \min\left\{\frac{S(nt)}{\sigma\sqrt{n}} : t \in [0, 1]\right\} \\ &= \frac{1}{\sigma\sqrt{n}} + \max\left\{\sigma^{-1}\hat{S}_n(t) : t \in [0, 1]\right\} - \min\left\{\sigma^{-1}\hat{S}_n(t) : t \in [0, 1]\right\} \end{aligned} \tag{4}$$

for every $n \in \mathbb{N}$. At this point, we define a functional $\varphi : C([0, 1], \mathbb{R}) \to \mathbb{R}$ by

$$\varphi(f(\cdot)) := \max\{f(t) : t \in [0, 1]\} - \min\{f(t) : t \in [0, 1]\}, \ \forall f(\cdot) \in C([0, 1], \mathbb{R}).$$

We claim that the map $\varphi : C([0,1],\mathbb{R}) \to \mathbb{R}$ is continuous under the uniform convergence topology, *i.e.*, the metric topology on $C([0,1],\mathbb{R})$ induced by the sup-norm. We recall that the sup-norm $\|\cdot\|_{\sup} : C(X,\mathbb{R}) \to \mathbb{R}_+$ defined on the space of continuous functions from a compact topological space $X$ to $\mathbb{R}$ is defined by

$$\|f\|_{\sup} := \sup\{|f(x)| : x \in X\} = \max\{|f(x)| : x \in X\}.$$

In order to prove this claim, given any $f(\cdot), g(\cdot) \in C([0,1],\mathbb{R})$, we may observe that

$$f(t) = g(t) + \{f(t) - g(t)\} \leq g(t) + \|g - f\|_{\sup} \leq \max\{g(t) : t \in [0,1]\} + \|g - f\|_{\sup}, \ \forall t \in [0,1],$$

thereby we arrive at

$$\max\{f(t) : t \in [0,1]\} \leq \max\{g(t) : t \in [0,1]\} + \|g - f\|_{\sup}, \ \forall t \in [0,1]. \tag{5}$$

On the other hand, one can see that

$$f(t) = g(t) + \{f(t) - g(t)\} \geq g(t) - \|g - f\|_{\sup} \geq \min\{g(t) : t \in [0,1]\} - \|g - f\|_{\sup}, \ \forall t \in [0,1],$$

thereby we reach

$$\min\{f(t) : t \in [0,1]\} \geq \min\{g(t) : t \in [0,1]\} - \|g - f\|_{\sup}, \ \forall t \in [0,1]. \tag{6}$$

Taking two pieces (5) and (6) collectively yields

$$\begin{aligned} \varphi(f(\cdot)) &= \max\{f(t) : t \in [0,1]\} - \min\{f(t) : t \in [0,1]\} \\ &\leq \max\{g(t) : t \in [0,1]\} - \min\{g(t) : t \in [0,1]\} + 2\|g - f\|_{\sup} \\ &= \varphi(g(\cdot)) + 2\|g - f\|_{\sup}. \end{aligned} \tag{7}$$

By interchanging the roles of $f(\cdot)$ and $g(\cdot)$ in the inequality (7), we obtain

$$\varphi(g(\cdot)) \leq \varphi(f(\cdot)) + 2\|g - f\|_{\sup}. \tag{8}$$

So combining two bounds (7) and (8) produces

$$|\varphi(g(\cdot)) - \varphi(f(\cdot))| \leq 2\|g - f\|_{\sup}, \ \forall f(\cdot), g(\cdot) \in C([0,1],\mathbb{R}). \tag{9}$$

From the inequality (9), we see that the function $\varphi : C([0,1],\mathbb{R}) \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant 2. So in particular, the function $\varphi : C([0,1],\mathbb{R}) \to \mathbb{R}$ is continuous under the uniform convergence topology on $C([0,1],\mathbb{R})$, *i.e.*, the metric topology on $C([0,1],\mathbb{R})$ induced by the sup-norm. Due to *Theorem 8.1.5* in [1], a simple consequence of the Donsker's invariance principle (*Theorem 8.1.4* in [1]) together with the continuous mapping theorem (*Theorem 3.2.10* in [1]), we see from $\mathbb{P}\{\pi_1\{B(\cdot)\} \in \mathcal{D}(\varphi)\} = 0$, where

$$\mathcal{D}(\varphi) := \{f(\cdot) \in C([0,1],\mathbb{R}) : \varphi : C([0,1],\mathbb{R}) \to \mathbb{R} \text{ is not continuous at } f(\cdot)\}$$

denotes the discontinuity set of $\varphi(\cdot)$, that

$$\varphi\left\{\pi_1\left(\frac{\hat{S}_n(\cdot)}{\sigma}\right)\right\} \xrightarrow{d} \varphi\{\pi_1(B(\cdot))\} \quad \text{as } n \to \infty. \tag{10}$$

Here, we note that $\mathcal{D}(\varphi) = \varnothing$. Hence,

$$
\begin{aligned}
\frac{R_n}{\sqrt{n}} &= \sigma\left(\frac{R_n}{\sigma\sqrt{n}} - \frac{1}{\sigma\sqrt{n}}\right) + \frac{1}{\sqrt{n}} \\
&\overset{\text{(b)}}{=} \sigma \cdot \varphi\left\{\pi_1\left(\frac{\hat{S}_n(\cdot)}{\sigma}\right)\right\} + \frac{1}{\sqrt{n}} \\
&\xrightarrow[d]{\text{(c)}} \sigma \cdot \{\pi_1\left(B(\cdot)\right)\} \\
&= \sigma\left(\max\{B(t) : t \in [0,1]\} - \min\{B(t) : t \in [0,1]\}\right),
\end{aligned}
$$

where the step (b) follows from (4), and the step (c) makes use of the result (10) together with the converging together lemma (*Exercise 3.2.13* in [1]). This establishes our desired result!

## References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.