MAS651 Theory of Stochastic Processes
Homework #1

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Throughout this homework, let $\mathbb{Z}_+$ denote the set of all non-negative integers, and $[a:b] := \{a, a+1, \cdots, b-1, b\}$ for $a, b \in \mathbb{Z}$ with $a \leq b$. We also write $[n] := [1:n]$ for $n \in \mathbb{N}$. Moreover, $\biguplus$ denotes the *disjoint union*, and given a set A and $k \in \mathbb{Z}_+$, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$. For instance, for $N \in \mathbb{N}$ and $i \in [0:N]$, $\binom{[N]}{i}$ denotes the set of all subsets of $[N]$ of size i.

Problem 1 (*Exercise 5.1.1.* in [1]).

Given a time step $n \in \mathbb{Z}_+$, we compute the conditional probability $\mathbb{P}\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\}$ for $i_0, i_1, \cdots, i_{n-1}, i, j \in \mathbb{S} := [0:N]$. We note that

$$X_{n+1} = |\{\xi_1, \xi_2, \cdots, \xi_{n+1}\}| = \begin{cases} |\{\xi_1, \xi_2, \cdots, \xi_n\}| = X_n & \text{if } \xi_{n+1} \in \{\xi_1, \xi_2, \cdots, \xi_n\}; \\ |\{\xi_1, \xi_2, \cdots, \xi_n\}| + 1 = X_n + 1 & \text{otherwise.} \end{cases} \tag{1}$$

So, $\mathbb{P}\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\} = 0 = \mathbb{P}\{X_{n+1} = j | X_n = i\}$ for all $j \in \mathbb{S} \setminus \{i, i+1\}$.

Now, we first consider the case $i < N$. Then, we have

$$
\begin{aligned}
& \mathbb{P}\left\{X_{n+1} = i+1 | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\} \\
= & \mathbb{P}\left\{\xi_{n+1} \in \mathbb{S} \setminus \{\xi_1, \cdots, \xi_n\} | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\} \\
\overset{\text{(a)}}{=} & \sum_{T \in \binom{[N]}{i}} \mathbb{P}\left\{\xi_{n+1} \in \mathbb{S} \setminus T, \{\xi_1, \cdots, \xi_n\} = T | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\} \\
= & \sum_{T \in \binom{[N]}{i}} \mathbb{P}\left\{\xi_{n+1} \in \mathbb{S} \setminus T | \{\xi_1, \cdots, \xi_n\} = T, X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\} \\
& \cdot \mathbb{P}\left\{\{\xi_1, \cdots, \xi_n\} = T | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\} \\
\overset{\text{(b)}}{=} & \sum_{T \in \binom{[N]}{i}} \underbrace{\mathbb{P}\left\{\xi_{n+1} \in \mathbb{S} \setminus T\right\}}_{= \frac{N-i}{N}} \mathbb{P}\left\{\{\xi_1, \cdots, \xi_n\} = T | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\} \\
\overset{\text{(c)}}{=} & \frac{N-i}{N} \\
\overset{\text{(d)}}{=} & \sum_{T \in \binom{[N]}{i}} \mathbb{P}\left\{\xi_{n+1} \in \mathbb{S} \setminus T\right\} \mathbb{P}\left\{\{\xi_1, \cdots, \xi_n\} = T | X_n = i\right\} \\
\overset{\text{(e)}}{=} & \sum_{T \in \binom{[N]}{i}} \mathbb{P}\left\{\xi_{n+1} \in \mathbb{S} \setminus T | \{\xi_1, \cdots, \xi_n\} = T, X_n = i\right\} \mathbb{P}\left\{\{\xi_1, \cdots, \xi_n\} = T | X_n = i\right\} \\
\overset{\text{(f)}}{=} & \mathbb{P}\left\{\xi_{n+1} \in \mathbb{S} \setminus \{\xi_1, \cdots, \xi_n\} | X_n = i\right\} \\
= & \mathbb{P}\left\{X_{n+1} = i+1 | X_n = i\right\},
\end{aligned}
\tag{2}
$$

where the step (a) follows from the relation

$$
\begin{aligned}
& \left\{\xi_{n+1} \in \mathbb{S} \setminus \{\xi_1, \cdots, \xi_n\}, X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\} \\
= & \biguplus_{T \in \binom{[n]}{i}} \left\{\xi_{n+1} \in \mathbb{S} \setminus T, \{\xi_1, \cdots, \xi_n\} = T, X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right\},
\end{aligned}
$$

the step (b) and the step (e) is due to the independence between ξ_{n+1} and the σ-field $\mathcal{F}_n^{\xi} := \sigma(\xi_1, \xi_2, \cdots, \xi_n)$ together with the fact that $X_1, X_2, \cdots, X_n$ are $\mathcal{F}_n^{\xi}$-measurable, the step (c) holds since $\xi_{n+1} \sim \mathrm{Unif}([N])$ and

$$
\{X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_n = i\} = \biguplus_{T \in \binom{[N]}{i}} \left\{X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_n = i, \{\xi_1, \cdots, \xi_n\} = T\right\},
$$

the step (d) is owing to the relation

$$
\{X_n = i\} = \biguplus_{T \in \binom{[N]}{i}} \left\{X_n = i, \{\xi_1, \cdots, \xi_n\} = T\right\}
$$

together with the assumption $\xi_{n+1} \sim \mathrm{Unif}([N])$, and finally the step (f) comes from the relation

$$
\left\{\xi_{n+1} \in \mathbb{S} \setminus \{\xi_1, \cdots, \xi_n\}, X_n = i\right\} = \biguplus_{T \in \binom{[N]}{i}} \left\{\xi_{n+1} \in \mathbb{S} \setminus T, \{\xi_1, \cdots, \xi_n\} = T, X_n = i\right\}.
$$

As an immediate consequence, one has

$$\begin{aligned}
&\mathbb{P}\left\{X_{n+1}=i \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\} \\
\stackrel{\text{(g)}}{=}\ & 1-\mathbb{P}\left\{X_{n+1}=i+1 \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\} \\
\stackrel{\text{(h)}}{=}\ & 1-\mathbb{P}\left\{X_{n+1}=i+1 \mid X_n=i\right\} \\
\stackrel{\text{(i)}}{=}\ & \mathbb{P}\left\{X_{n+1}=i \mid X_n=i\right\} \\
\stackrel{\text{(j)}}{=}\ & 1-\frac{N-i}{N}=\frac{i}{N},
\end{aligned} \tag{3}$$

where the step (g) and (i) follows from the fact that $X_{n+1} \in \{i, i+1\}$ given that $X_n = i$, and the step (h) and (j) comes from the computation (2). Hence, we eventually obtain

$$\mathbb{P}\left\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\}=\mathbb{P}\left\{X_{n+1}=j \mid X_n=i\right\}= \begin{cases}\frac{N-i}{N} & \text{if } j=i+1; \\ \frac{i}{N} & \text{if } j=i; \\ 0 & \text{otherwise,}\end{cases} \tag{4}$$

when $i < N$.

For the remaining case $i = N$, it's clear that

$$\mathbb{P}\left\{X_{n+1}=j \mid X_n=N, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\}=\mathbb{P}\left\{X_{n+1}=j \mid X_n=N\right\}= \begin{cases}1 & \text{if } j=N; \\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

In particular, we can see from (4) and (5) that

$$\mathbb{P}\left\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\}=\mathbb{P}\left\{X_{n+1}=j \mid X_n=i\right\}$$

for all $i_0, \cdots, i_{n-1}, i, j \in \mathbb{S}$, thereby the $\mathbb{S}$-valued stochastic process $\{X_n\}_{n=0}^{\infty}$ is a Markov chain according to the definition of Markov chains with countable state space in *Section 5.1* of [1]. Finally, its transition probability $p : \mathbb{S} \times \mathbb{S} \to [0, 1]$ is obtained immediately from (4) and (5) as

$$p(i, j)=\mathbb{P}\left\{X_{n+1}=j \mid X_n=i\right\}= \begin{cases}\frac{i}{N} & \text{if } j=i; \\ \frac{N-i}{N} & \text{if } j=i+1 \text{ and } i<N; \\ 0 & \text{otherwise.}\end{cases}$$

Problem 2 (*Exercise 5.1.2.* in [1]).

Assume on the contrary that $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with the countable state space $\mathbb{S} := \mathbb{Z}$. According to the definition of Markov chains with countable state space in *Section 5.1* in [1], $\{X_n\}_{n=0}^{\infty}$ obeys the *Markov property* for every time step $n \in \mathbb{Z}_+$:

$$\mathbb{P}\left\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\}=\mathbb{P}\left\{X_{n+1}=j \mid X_n=i\right\} \tag{6}$$

for all $i_0, \cdots, i_{n-1}, i, j \in \mathbb{S}$. In particular, the value of the probability $\mathbb{P}\left\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\}$ should be irrelevant of the path of past states $(i_0, i_1, \cdots, i_{n-1}) \in \mathbb{S}^n$. Now, we consider the fourth time step, *i.e.*, $n = 4$. One can consider the following two paths of the stochastic process $\{X_n\}_{n=0}^{\infty}$ between the times

0 and 4: $(X_0, X_1, X_2, X_3, X_4) = (0, 1, 1, 1, 2)$ and $(X_0, X_1, X_2, X_3, X_4) = (0, 0, 0, 1, 2)$. Due to the Markov property (6), we have

$$\begin{aligned}\mathbb{P}\{X_4 = 2|X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0\} &= \mathbb{P}\{X_4 = 2|X_3 = 1\} \\ &= \mathbb{P}\{X_4 = 2|X_3 = 1, X_2 = 0, X_1 = 0, X_0 = 0\}.\end{aligned} \tag{7}$$

1. We can easily see that $(X_0, X_1, X_2, X_3, X_4) = (0, 1, 1, 1, 2)$ if and only if $(\xi_1, \xi_2, \xi_3, \xi_4) = (+1, -1, +1, +1)$. On the other hand, $(X_0, X_1, X_2, X_3) = (0, 1, 1, 1)$ if and only if $(\xi_1, \xi_2) = (+1, -1)$ and ξ_3 can attain any value in $\{-1, +1\}$. Therefore, we conclude that

$$\begin{aligned}\{(X_0, X_1, X_2, X_3, X_4) = (0, 1, 1, 1, 2)\} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) = (+1, -1, +1, +1)\}; \\ \{(X_0, X_1, X_2, X_3) = (0, 1, 1, 1)\} &= \{(\xi_1, \xi_2) = (+1, -1)\},\end{aligned}$$

thereby $\mathbb{P}\{(X_0, X_1, X_2, X_3, X_4) = (0, 1, 1, 1, 2)\} = \left(\frac{1}{2}\right)^4$ and $\mathbb{P}\{(X_0, X_1, X_2, X_3) = (0, 1, 1, 1)\} = \left(\frac{1}{2}\right)^2$. Hence, we obtain

$$\mathbb{P}\{X_4 = 2|X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0\} = \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)^2} = \frac{1}{4}. \tag{8}$$

2. It's clear that $(X_0, X_1, X_2, X_3, X_4) = (0, 0, 0, 1, 2)$ if and only if $(\xi_1, \xi_2, \xi_3, \xi_4) = (-1, +1, +1, +1)$, and $(X_0, X_1, X_2, X_3) = (0, 0, 0, 1)$ if and only if $(\xi_1, \xi_2, \xi_3) = (-1, +1, +1)$. Thus, we deduce that

$$\begin{aligned}\{(X_0, X_1, X_2, X_3, X_4) = (0, 0, 0, 1, 2)\} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) = (-1, +1, +1, +1)\}; \\ \{(X_0, X_1, X_2, X_3) = (0, 0, 0, 1)\} &= \{(\xi_1, \xi_2, \xi_3) = (-1, +1, +1)\},\end{aligned}$$

thereby $\mathbb{P}\{(X_0, X_1, X_2, X_3, X_4) = (0, 0, 0, 1, 2)\} = \left(\frac{1}{2}\right)^4$ and $\mathbb{P}\{(X_0, X_1, X_2, X_3) = (0, 0, 0, 1)\} = \left(\frac{1}{2}\right)^3$. So, one has

$$\mathbb{P}\{X_4 = 2|X_3 = 1, X_2 = 0, X_1 = 0, X_0 = 0\} = \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)^3} = \frac{1}{2}. \tag{9}$$

Combining the above two straightforward computations, we can see that the equation (7) does not hold, which gives a contradiction to our initial assumption. Hence, the stochastic process $\{X_n\}_{n=0}^{\infty}$ is not a Markov chain.

Problem 3 (*Exercise 5.1.5.* in [1]: Bernoulli-Laplace model of a diffusion)**.**

Let X_n denote the number of black balls in the left urn at the n-th time step. Let us take a closer look at all possible outcomes of X_{n+1} given that $X_n = i$, where $i \in \mathbb{S} := [0 : b]$. If $0 < i < b$, there are three possible outcomes for X_{n+1}, $X_{n+1} \in \{i-1, i, i+1\}$, with the corresponding choice of balls from each urn at the $(n+1)$-th time step, and the choices can be described as follows:

$$\begin{aligned}X_{n+1} = i+1 &\Leftrightarrow \text{We pick a white ball in the left urn, and a black ball in the right urn.} \\ X_{n+1} = i &\Leftrightarrow \text{We pick balls of the same color in each urn.} \\ X_{n+1} = i-1 &\Leftrightarrow \text{We pick a black ball in the left urn, and a white ball in the right urn.}\end{aligned} \tag{10}$$

If $i = 0$, there are two possible outcomes for X_{n+1}, $X_{n+1} \in \{0, 1\}$, with the corresponding choice of balls from each urn at the $(n+1)$-th time step, and the choices can be described as follows:

$$\begin{aligned}X_{n+1} = 1 &\Leftrightarrow \text{We pick a black ball in the right urn.} \\ X_{n+1} = 0 &\Leftrightarrow \text{We pick a white ball in the right urn.}\end{aligned} \tag{11}$$

Finally, if $i = b$, there are also two possible outcomes for X_{n+1}, $X_{n+1} \in \{b-1, b\}$, with the corresponding choice of balls from each urn at the $(n+1)$-th time step, and the choices can be described as follows:

$$\begin{aligned} X_{n+1} = b &\Leftrightarrow \text{We pick a white ball in the left urn.} \\ X_{n+1} = b-1 &\Leftrightarrow \text{We pick a black ball in the left urn.} \end{aligned} \tag{12}$$

Before we compute the transition probability $p : \mathbb{S} \times \mathbb{S} \to [0,1]$ of the $\mathbb{S}$-valued Markov chain $\{X_n\}_{n=0}^{\infty}$, we note that $\{X_n\}_{n=0}^{\infty}$ is indeed a *Markov chain* since the value of X_{n+1} is determined solely based on the value of X_n and the choice of balls from each urn at the $(n+1)$-th time step. This means that the past history between the times 0 and $n-1$ has no effect on the value of X_{n+1}, thereby the stochastic process $\{X_n\}_{n=0}^{\infty}$ satisfies the Markov property. According to the definition of Markov chains with countable state space in *Section 5.1* in [1], it is a Markov chain with countable state space. Now, we compute the transition probability of $\{X_n\}_{n=0}^{\infty}$. At this point, we note that given $X_n = i$,

$$\begin{aligned} &\text{(the probability to choose a black ball in the left urn at the } (n+1)\text{-th time step.)} = \frac{i}{m}; \\ &\text{(the probability to choose a white ball in the left urn at the } (n+1)\text{-th time step.)} = 1 - \frac{i}{m}; \\ &\text{(the probability to choose a black ball in the right urn at the } (n+1)\text{-th time step.)} = \frac{b-i}{m}; \\ &\text{(the probability to choose a white ball in the right urn at the } (n+1)\text{-th time step.)} = 1 - \frac{b-i}{m}. \end{aligned} \tag{13}$$

1. If $0 < i < b$, then we can see from (10) and (13) that

$$p(i,j) = \mathbb{P}\{X_{n+1} = j | X_n = i\} = \begin{cases} \frac{b-i}{m}\left(1 - \frac{i}{m}\right) & \text{if } j = i+1; \\ \frac{i}{m} \cdot \frac{b-i}{m} + \left(1 - \frac{i}{m}\right)\left(1 - \frac{b-i}{m}\right) & \text{if } j = i; \\ \frac{i}{m}\left(1 - \frac{b-i}{m}\right) & \text{if } j = i-1; \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

2. If $i = 0$, then we can deduce from (11) and (13) that

$$p(0,j) = \mathbb{P}\{X_{n+1} = j | X_n = 0\} = \begin{cases} 1 - \frac{b}{m} & \text{if } j = 0; \\ \frac{b}{m} & \text{if } j = 1; \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

3. If $i = b$, then we can conclude from (12) and (13) that

$$p(b,j) = \mathbb{P}\{X_{n+1} = j | X_n = b\} = \begin{cases} 1 - \frac{b}{m} & \text{if } j = b; \\ \frac{b}{m} & \text{if } j = b-1; \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Combining all of the above computations (14)–(16) provides the following succinct form of the transition probability $p : \mathbb{S} \times \mathbb{S} \to [0,1]$ of the Markov chain $\{X_n\}_{n=0}^{\infty}$: for any $(i,j) \in \mathbb{S} \times \mathbb{S}$,

$$p(i,j) = \begin{cases} \frac{b-i}{m}\left(1 - \frac{i}{m}\right) & \text{if } j = i+1 \text{ and } i < b; \\ \frac{i}{m} \cdot \frac{b-i}{m} + \left(1 - \frac{i}{m}\right)\left(1 - \frac{b-i}{m}\right) & \text{if } j = i; \\ \frac{i}{m}\left(1 - \frac{b-i}{m}\right) & \text{if } j = i-1 \text{ and } i > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4 (*Exercise 5.1.6.* in [1]).

To begin with, we use the symbol Θ instead of θ. Then, we have mutually independent and identically distributed with uniform distribution over $(0,1)$, Θ and $\{U_n : n \in \mathbb{N}\}$. Also, $X_i := +1$ if $U_i \le \Theta$; $X_i := -1$ otherwise, $S_n := \sum_{i=1}^n X_i$ for $n \in \mathbb{N}$, and $S_0 := 0$.

Now, we compute the conditional probability $\mathbb{P}\{X_{n+1} = 1 | X_1, \cdots, X_n\}$. The joint probability density function of the $(n+1)$-dimensional random vector $(\Theta, U_1, \cdots, U_n)$, $f_n : \mathbb{R}^{n+1} \to [0, +\infty)$, is given by

$$f_n(\theta, u_1, \cdots, u_n) = \mathbb{1}_{(0,1)}(\theta)\left[\prod_{i=1}^n \mathbb{1}_{(0,1)}(u_i)\right], \ \forall (\theta, u_1, \cdots, u_n) \in \mathbb{R}^{n+1},$$

from the independence of $\{\Theta, U_1, \cdots, U_n\}$. We can easily observe that $X_i = x_i$ if and only if $\text{sign}(\Theta - U_i) = x_i$ for $x_i \in \{-1, +1\}$, where $\text{sign}(x) := +1$ if $x \ge 0$, and $\text{sign}(x) := -1$ otherwise. Then, we have

$$\begin{aligned} \mathbb{P}\{X_1 = x_1, \cdots, X_n = x_n\} &= \mathbb{P}\{\text{sign}(\Theta - U_1) = x_1, \cdots, \text{sign}(\Theta - U_n) = x_n\} \\ &= \int_{\left\{(\theta, u_1, \cdots, u_n) \in (0,1)^{n+1} :\ \text{sign}(\theta - u_i) = x_i,\ \forall i \in [n]\right\}} \mathrm{d}\theta \mathrm{d}u_1 \cdots \mathrm{d}u_n \\ &= \int_0^1 \left[\int_{\Omega(x_1, \cdots, x_n; \theta)} \mathrm{d}u_1 \cdots \mathrm{d}u_n\right] \mathrm{d}\theta \\ &= \int_0^1 \lambda_n\{\Omega(x_1, \cdots, x_n; \theta)\} \mathrm{d}\theta, \end{aligned} \tag{17}$$

where $\Omega(x_1, \cdots, x_n; \theta) := \{(u_1, \cdots, u_n) \in (0,1)^n : \text{sign}(\theta - u_i) = x_i,\ \forall i \in [n]\} \subseteq (0,1)^n$. Here, $\lambda_n(\cdot)$ refers to the standard Lebesgue measure on $\mathbb{R}^n$. For $i \in [n]$, let

$$\mathcal{I}(x_i; \theta) := \begin{cases} (0, \theta] & \text{if } x_i = +1; \\ (\theta, 1) & \text{otherwise.} \end{cases}$$

Then, it's clear that $\Omega(x_1, \cdots, x_n; \theta) = \prod_{i=1}^n \mathcal{I}(x_i; \theta)$, thereby

$$\begin{aligned} \lambda_n\{\Omega(x_1, \cdots, x_n; \theta)\} &= \prod_{i=1}^n \lambda_1(\mathcal{I}(x_i; \theta)) \\ &= \prod_{i=1}^n \left(\theta^{\frac{1+x_i}{2}} (1-\theta)^{\frac{1-x_i}{2}}\right) \\ &= \theta^{\frac{n + \sum_{i=1}^n x_i}{2}} (1-\theta)^{\frac{n - \sum_{i=1}^n x_i}{2}}. \end{aligned} \tag{18}$$

Putting (18) into (17) yields

$$\begin{aligned}\mathbb{P}\left\{X_1 = x_1, \cdots, X_n = x_n\right\} &= \int_0^1 \theta^{\frac{n+\sum_{i=1}^n x_i}{2}} (1-\theta)^{\frac{n-\sum_{i=1}^n x_i}{2}} \mathrm{d}\theta \\ &= \mathbf{B}\left(\frac{n+\sum_{i=1}^n x_i}{2}+1, \frac{n-\sum_{i=1}^n x_i}{2}+1\right) \\ &= \frac{\Gamma\left(\frac{n+\sum_{i=1}^n x_i}{2}+1\right)\Gamma\left(\frac{n-\sum_{i=1}^n x_i}{2}+1\right)}{\Gamma(n+2)} \\ &= \frac{1}{(n+1)!}\Gamma\left(\frac{n+\sum_{i=1}^n x_i}{2}+1\right)\Gamma\left(\frac{n-\sum_{i=1}^n x_i}{2}+1\right),\end{aligned} \tag{19}$$

where $\mathbf{B}(\cdot,\cdot) : (0,+\infty)\times(0,+\infty)\to(0,+\infty)$ denotes the *beta function* defined by

$$\mathbf{B}(\alpha,\beta) := \int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1}\mathrm{d}\theta.$$

It is well-known that $\mathbf{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ for $\alpha,\beta>0$, where $\Gamma(\cdot):(0,+\infty)\to(0,+\infty)$ is the *gamma function*. Therefore, we deduce from (19) that for any $x_1, x_2, \cdots, x_{n+1} \in \{-1,+1\}$,

$$\begin{aligned}\mathbb{P}\left\{X_{n+1} = x_{n+1} \middle| X_n = x_n, \cdots, X_1 = x_1\right\} &= \frac{\mathbb{P}\left\{X_1 = x_1, \cdots, X_n = x_n, X_{n+1} = x_{n+1}\right\}}{\mathbb{P}\left\{X_1 = x_1, \cdots, X_n = x_n\right\}} \\ &= \frac{\frac{1}{(n+2)!}\Gamma\left(\frac{n+s_n}{2}+\frac{1+x_{n+1}}{2}+1\right)\Gamma\left(\frac{n-s_n}{2}+\frac{1-x_{n+1}}{2}+1\right)}{\frac{1}{(n+1)!}\Gamma\left(\frac{n+s_n}{2}+1\right)\Gamma\left(\frac{n-s_n}{2}+1\right)} \\ &= \begin{cases}\frac{1}{2}+\frac{s_n}{2(n+2)} & \text{if } x_{n+1}=1; \\ \frac{1}{2}-\frac{s_n}{2(n+2)} & \text{otherwise.}\end{cases} \\ &= \frac{1}{2}+\frac{x_{n+1}\left(\sum_{i=1}^n x_i\right)}{2(n+2)},\end{aligned} \tag{20}$$

where $s_n := \sum_{i=1}^n x_i$. In particular, we have the desired computation

$$\mathbb{P}\left\{X_{n+1} = 1 \middle| X_n, \cdots, X_1\right\} = \frac{1}{2}+\frac{1}{2(n+2)}\sum_{i=1}^n X_i.$$

Finally, let's prove that the $\mathbb{Z}$-valued stochastic process $\{S_n\}_{n=0}^\infty$ is a temporally inhomogeneous Markov chain. For every $(s_1, s_2, \cdots, s_{n+1}) \in \mathbb{Z}^{n+1}$, we have

$$\begin{aligned}\mathbb{P}\left\{S_{n+1} = s_{n+1} \middle| S_n = s_n, \cdots, S_1 = s_1, S_0 = 0\right\} &= \mathbb{P}\left\{X_{n+1} = s_{n+1}-s_n \middle| X_n = s_n - s_{n-1}, \cdots, X_1 = s_1\right\} \\ &\overset{\text{(a)}}{=} \left\{\frac{1}{2}+\frac{(s_{n+1}-s_n)\,s_n}{2(n+2)}\right\}\mathbb{1}_{\{-1,+1\}}(s_{n+1}-s_n),\end{aligned} \tag{21}$$

where the step (a) comes from (20). Hence, we obtain

$$\begin{aligned}
&\mathbb{P}\left\{ S_{n+1} = s_{n+1} \middle| S_n = s_n \right\} \\
= &\sum_{(s_1,\cdots,s_{n-1})\in\mathbb{Z}^{n-1}} \mathbb{P}\left\{ S_{n+1} = s_{n+1} \middle| S_n = s_n, \cdots, S_1 = s_1, S_0 = 0 \right\} \mathbb{P}\left\{ S_0 = 0, S_1 = s_1, \cdots, S_{n-1} = s_{n-1} \middle| S_n = s_n \right\} \\
= &\sum_{(s_1,\cdots,s_{n-1})\in\mathbb{Z}^{n-1}} \underbrace{\left\{ \frac{1}{2} + \frac{(s_{n+1} - s_n)\, s_n}{2(n+2)} \right\} \mathbb{1}_{\{-1,+1\}}\left(s_{n+1} - s_n\right)}_{\text{independent of } (s_1,\cdots,s_{n-1})} \mathbb{P}\left\{ S_0 = 0, S_1 = s_1, \cdots, S_{n-1} = s_{n-1} \middle| S_n = s_n \right\} \\
= &\left\{ \frac{1}{2} + \frac{(s_{n+1} - s_n)\, s_n}{2(n+2)} \right\} \mathbb{1}_{\{-1,+1\}}\left(s_{n+1} - s_n\right) \\
= &\, \mathbb{P}\left\{ S_{n+1} = s_{n+1} \middle| S_n = s_n, \cdots, S_1 = s_1, S_0 = 0 \right\},
\end{aligned}$$

thereby $\{S_n\}_{n=0}^{\infty}$ is a $\mathbb{Z}$-valued Markov chain, and its transition probability at the n-th time step is given by

$$p_n(i,j) := \begin{cases} \frac{1}{2} + \frac{i}{2(n+2)} & \text{if } j = i+1; \\ \frac{1}{2} - \frac{i}{2(n+2)} & \text{if } j = i-1; \\ 0 & \text{otherwise.} \end{cases}$$

Since $p_n(\cdot,\cdot) : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ is not constant of $n \in \mathbb{Z}_+$, we can conclude that the Markov chain $\{S_n\}_{n=0}^{\infty}$ is temporally inhomogeneous.

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.