# KAIST 2021 Spring Semester
# MAS575 Combinatorics
# Final Report

**Jeonghwan Lee**

University: KAIST
Department: Department of Mathematical Sciences

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Throughout this article, we would like to explore the existence of *generalized $f$-factors* in finite simple graphs, where the notion of "generalized $f$-factors" was introduced by László Lovász [10].

## 1 Basic Notions

To begin with, we introduce the term *$f$-factors in finite simple graphs* with traditional meaning. We first provide you some significant notations.

**Notation 1.1.** Let $G$ be a graph.

1. $\mathcal{V}(G)$: the vertex set of $G$;
2. $\mathcal{E}(G)$: the edge set of $G$;
3. $\deg_G(v)$: the degree of the vertex $v \in \mathcal{V}(G)$ in $G$;
4. Given any function $f : \mathcal{V}(G) \to \mathbb{R}$ and subset $S \subseteq \mathcal{V}(G)$, let $f(S) := \sum_{v \in S} f(v)$;
5. $\mathsf{odd}(G)$: the number of odd component in $G$, *i.e.*, the number of components in $G$ of odd order;
6. Given any two subsets $S$ and $T$ of $\mathcal{V}(G)$ with $S \cap T = \varnothing$, let $q(S,T)$ denote the number of components $C$ of $G \setminus (S \cup T)$ such that $e_G(C,T) + f(\mathcal{V}(C))$ is odd, *i.e.*,

$$q(S,T) := \left| \{ C \text{ component in } G : e_G(C,T) + f(\mathcal{V}(C)) \equiv 1 \,(\mathrm{mod}\ 2) \} \right|.$$

**Definition 1.1** ($k$-factors in $G$)**.** Given a positive integer $k$, a *$k$-factor in the graph $G$* refers to a $k$-regular spanning subgraph of $G$.

**Definition 1.2** (A generalization of $k$-factors: $f$-factors in $G$ [13])**.** Given a function $f : \mathcal{V}(G) \to \mathbb{Z}_+$ with $0 \le f(v) \le \deg_G(v)$ for every $v \in \mathcal{V}(G)$, where $\mathbb{Z}_+$ is the set of all non-negative integers, an *$f$-factor in $G$* is a spanning subgraph $H$ of $G$ such that $\deg_H(v) = f(v)$ for all $v \in \mathcal{V}(H) = \mathcal{V}(G)$. Note that if we consider the function $f : \mathcal{V}(G) \to \mathbb{Z}_+$ defined by $f(v) = k$ for all $v \in \mathcal{V}(G)$, we see that Definition 1.2 recovers Definition 1.1.

**Definition 1.3** (A further generalization of $k$-factors: generalized $f$-factors in $G$ [10])**.** Given any function $f : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ assigning to each vertex $v \in \mathcal{V}(G)$ a set of integers $f(v) \subseteq [0 : \deg_G(v)] := \{0, 1, \cdots, \deg_G(v)\}$, where $2^{\mathbb{Z}}$ denotes the power set of the set of integers, a *generalized $f$-factor in $G$* is a spanning subgraph $H$ of $G$ such that $\deg_H(v) \in f(v)$ for all $v \in \mathcal{V}(H) = \mathcal{V}(G)$. We notice that if we consider the function $f : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ such that $|f(v)| = 1$ for all $v \in \mathcal{V}(G)$, Definition 1.3 retrieves Definition 1.2. A *partial generalized $f$-factor in $G$* refers to an $\tilde{f}$-factor in $G$, where $\tilde{f} : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ is given by $\tilde{f}(v) := f(v) \cup \{0\}$ for every $v \in \mathcal{V}(G)$.

## 2 Existing Works

As mentioned above, we are interested in the conditions for the existence of generalized $f$-factors in a given finite simple graph $G$. A necessary and sufficient condition for the existence of $f$-factors in $G$, *i.e.*, generalized $f$-factors in $G$ for the case $|f(v)| = 1$, $\forall v \in \mathcal{V}(G)$, is characterized by William T. Tutte [13]. The following well-known results in the field of graph theory provide such necessary and sufficient conditions.

**Theorem 2.1** (Tutte's 1-factor theorem [13])**.** *A finite simple graph $G$ has a 1-factor, i.e., $G$ has a perfect matching if and only if $\mathsf{odd}(G \setminus S) \le |S|$ for all $S \subseteq \mathcal{V}(G)$.*

**Theorem 2.2** (Tutte's $k$-factor theorem [13])**.** *Let $G$ be a finite simple graph and $f : \mathcal{V}(G) \to \mathbb{Z}_+$ be any function. Then $G$ has an $f$-factor if and only if*

$$q(S, T) - \sum_{v \in T} \deg_{G \setminus S}(v) \le f(S) - f(T)$$

*for all $S, T \subseteq \mathcal{V}(G)$ with $S \cap T = \varnothing$.*

Later in 1985, Richard P. Anstee gave an algorithmic proof of Theorem 2.2 when the given graph $G$ is a finite multi-graph, which also provides an algorithm that either finds an $f$-factor in $G$ or shows that it does not exist within $\mathcal{O}\left(|\mathcal{V}(G)|^3\right)$ operations [5].

However, if we allow the function $f : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ with $f(v) \subseteq [0 : \deg_G(v)]$ for every $v \in \mathcal{V}(G)$ to be $|f(v)| \ge 2$ for all $v \in \mathcal{V}(G)$, even when $|f(v)| = 2$ for all $v \in \mathcal{V}(G)$, there are no *necessary and sufficient condition* for the existence of a generalized $f$-factor in $G$. This is because the decision problem of determining whether a given simple finite graph has a generalized $f$-factor is known to be algorithmically hard in general for this case [10]. More precisely, Lovász proved in his paper [10] that if there is a vertex $v \in \mathcal{V}(G)$ such that the corresponding set $f(v) \subseteq [0 : \deg_G(v)]$ contains two consecutive integers differ by more than 2, then the decision problem is NP-complete. Nevertheless, he was able to provide in his papers [9] and [10] a necessary and sufficient condition for the existence of a generalized $f$-factor in a given simple finite graph $G$ whenever for every $v \in \mathcal{V}(G)$, $f(v)$ consists of consecutive integers, and no two consecutive integers in $f(v)$ differ by more than two, respectively.

In our main reference [12], the authors provide one sufficient condition for the existence of a *generalized $f$-factor* in a given finite simple graph $G$ and one sufficient condition for the existence of a *partial generalized $f$-factor* in $G$. Their results are known to be best possible, and the *combinatorial nullstellensatz* [3] plays a crucial role in their proofs. Later, Frank, Lau, and Szabó [7] found an elementary and constructive proof for the main results presented in [12] without using the combinatorial nullstellensatz. More precisely, [7] proved the finite simple graph version of our new extension to the case of hypergraphs. See Theorem 4.1 for further details. Moreover, the proof of [7] gives rise to a polynomial-time algorithm for finding a generalized $f$-factor in graphs. Recently, the paper [2] established a sufficient condition for the presence of a $k$-edge-connected generalized $f$-factor in $2k$-edge-connected simple finite graphs for every non-negative integer $k$. This result is a generalization of the aforementioned main result in [12]: the case $k = 0$.

On the other hand, there exist several results regarding the existence of graph factors along another line. Lu, Wang, and Yu [11] investigated the existence of generalized $f$-factors in regular graphs. In order to check the detailed discussion, we refer to *Theorem 1.11* therein. Lately, the authors of the paper [1] discovered a tight sufficient condition for the $f$-avoiding property of finite simple bipartite graphs by following the proof presented by [7] of the main result of [12]. For the self-containedness, we provide the definition of the $f$-avoiding property of a finite simple graph $G$, where $f : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ is a given function: we say that $G$ is $f$-avoiding if there is an orientation $D : \mathcal{E}(G) \to \mathcal{V}(G)$ of $G$ such that $\deg_D^+(v) \in [0 : \deg_G(v)] \setminus f(v)$ for every $v \in \mathcal{V}(G)$, where $\deg_D^+(v)$ denotes the out-degree of $v$ in the directed graph $G$ with respect to the orientation $D$.

## 3 Main Results

We first state our main tool, which is known as the *combinatorial nullstellensatz* [3]:

**Theorem 3.1** (The combinatorial nullstellensatz [3])**.** *Let $\mathbb{F}$ be a field and $f(x_1, x_2, \cdots, x_n) \in \mathbb{F}[x_1, x_2, \cdots, x_n]$. Suppose that $\deg(f) = d = \sum_{i=1}^n d_i$ and the coefficient of $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ in $f(x_1, x_2, \cdots, x_n)$ is non-zero. If $L_1, L_2, \cdots, L_n \subseteq \mathbb{F}$ such that $|L_j| > d_j$ for all $j \in [n]$, then there is a vector $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \prod_{j=1}^n L_j$ such that $f(\mathbf{a}) \neq 0$.*

The following result provides a sufficient condition for the presence of generalized $f$-factors in a finite simple, connected graph $G$:

**Theorem 3.2** ([12])**.** *Let $G$ denote a finite simple, connected graph, and $f : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ be a function such that $f(v) \subseteq [0 : \deg_G(v)]$ for every $v \in \mathcal{V}(G)$. Suppose that*

$$|f(v)| > \left\lceil \frac{\deg_G(v)}{2} \right\rceil \tag{1}$$

*for every $v \in \mathcal{V}(G)$. Then, $G$ has a generalized $f$-factor.*

*Proof of Theorem 3.2.*

We consider a polynomial $g(\mathbf{x})$ over $\mathbb{R}$ with $|\mathcal{E}(G)|$ variables $x_e$, $e \in \mathcal{E}(G)$, defined by

$$g(\mathbf{x}) := \prod_{v \in \mathcal{V}(G)} \left[ \prod_{c \in [0:\deg_G(v)] \setminus f(v)} \left\{ \left( \sum_{e \in \mathcal{E}(G)} \mathbb{I}_G(v, e) \cdot x_e \right) - c \right\} \right], \ \forall \mathbf{x} = (x_e : e \in \mathcal{E}(G)) \in \mathbb{R}^{\mathcal{E}(G)},$$

where $\mathbb{I}_G(\cdot,\cdot) : \mathcal{V}(G) \times \mathcal{E}(G) \to \{0,1\}$ denotes the *incidence matrix of* $G$. It's clear that

$$\deg(g) = \sum_{v \in \mathcal{V}(G)} |[0 : \deg_G(v)] \setminus f(v)| \,.$$

Let $L_e := \{0,1\}$ for each $e \in \mathcal{E}(G)$. In order to employ the combinatorial nullstellensatz, we should prove that there exists a monomial $\prod_{e \in \mathcal{E}(G)} x_e^{t_e}$ with non-zero coefficient in the expansion of $g(\mathbf{x})$ such that

$$t_e \in \{0,1\}\,,\ \forall e \in \mathcal{E}(G), \quad \text{and} \quad \sum_{e \in \mathcal{E}(G)} t_e = \deg(g) = \sum_{v \in \mathcal{V}(G)} |[0 : \deg_G(v)] \setminus f(v)| \,.$$

**Lemma 3.1.** *There exists an orientation $D : \mathcal{E}(G) \to \mathcal{V}(G)$ assigning to each edge $e \in \mathcal{E}(G)$ the head node $D(e) \in e$ of the edge $e$ such that*

$$\left\lfloor \frac{\deg_G(v)}{2} \right\rfloor \le \deg_D^+(v) \le \left\lceil \frac{\deg_G(v)}{2} \right\rceil, \ \forall v \in \mathcal{V}(G), \tag{2}$$

*where $\deg_D^+(v)$ denotes the out-degree of $v \in \mathcal{V}(G)$ under the orientation $D$.*

*Proof of Lemma 3.1.*

We may observe that the original graph $G$ may not have an Euler tour. In order to address this issue, we add a new vertex $v^*$ to the original graph $G$ and connect it to all vertices in $\mathcal{V}(G)$ of odd degrees. More formally, let $G^*$ be the new finite simple, connected graph defined by

$$\mathcal{V}(G^*) := \mathcal{V}(G) \cup \{v^*\}\,, \quad \text{and} \quad \mathcal{E}(G^*) := \mathcal{E}(G) \cup \{\{v^*, w\} : w \in \mathcal{V}_{\text{odd}}(G)\}\,,$$

where $\mathcal{V}_{\text{odd}}(G)$ denotes the set of all vertices in $\mathcal{V}(G)$ of odd degrees. Then it's clear that every vertex of $G^*$ has even degree. Due to Euler's theorem, the augmented graph $G^*$ has an Euler tour. Let $D^* : \mathcal{E}(G^*) \to \mathcal{V}(G^*)$ denote the Eulerian orientation associated to the Euler tour in $G^*$, *i.e.*, the orientation on $G^*$ assigning to each edge $e \in \mathcal{E}(G^*)$ the head node $D^*(e) \in e$ of the edge $e$ obtained by orienting the edges according to the Euler tour in $G^*$. Then we have $\deg_{D^*}^+(v) = \deg_{D^*}^-(v) = \frac{1}{2}\deg_{G^*}(v)$ for every $v \in \mathcal{V}(G^*)$. We define $D := D^*|_{\mathcal{E}(G)} : \mathcal{E}(G) \to \mathcal{V}(G)$. It's clear that $D$ is an orientation on the original graph $G$. Moreover, one can see that if $v \in \mathcal{V}_{\text{even}}(G)$, where $\mathcal{V}_{\text{even}}(G)$ refers to the set of all vertices in $\mathcal{V}(G)$ of even degrees, then

$$\deg_D^+(v) = \deg_{D^*}^+(v) = \frac{1}{2}\deg_{G^*}(v) = \frac{1}{2}\deg_G(v), \tag{3}$$

while if $v \in \mathcal{V}_{\text{odd}}(G)$, then

$$\begin{aligned} \deg_D^+(v) &\ge \deg_{D^*}^+(v) - 1 = \frac{1}{2}\deg_{G^*}(v) - 1 = \frac{1}{2}\{\deg_G(v)+1\} - 1 = \frac{1}{2}\{\deg_G(v)-1\}\,; \\ \deg_D^+(v) &\le \deg_{D^*}^+(v) = \frac{1}{2}\deg_{G^*}(v) = \frac{1}{2}\{\deg_G(v)+1\}\,. \end{aligned} \tag{4}$$

Taking two pieces (3) and (4) collectively yields

$$\left\lfloor \frac{\deg_G(v)}{2} \right\rfloor \le \deg_D^+(v) \le \left\lceil \frac{\deg_G(v)}{2} \right\rceil, \ \forall v \in \mathcal{V}(G),$$

as desired.

□

We consider the following sets of edges in $\mathcal{E}(G)$ which are incident to a vertex $v \in \mathcal{V}(G)$, where

$$\mathcal{I}_D(v) := \{e \in \mathcal{E}(G) : v \in e \text{ and } D(e) = v\}\,;$$
$$\mathcal{O}_D(v) := \{e \in \mathcal{E}(G) : v \in e \text{ and } D(e) \neq v\}\,,$$

where $D : \mathcal{E}(G) \to \mathcal{V}(G)$ refers to an orientation on $G$ constructed in Lemma 3.1. In words, $\mathcal{I}_D(v)$ denotes the set of all edges in $\mathcal{E}(G)$ incident to $v$ such that they have $v$ as their head node under the orientation $D$, while $\mathcal{O}_D(v)$ denotes the set of all edges in $\mathcal{E}(G)$ incident to $v$ such that they have $v$ as their tail node under the orientation $D$. From their definition, it's clear that $\deg_D^+(v) = |\mathcal{O}_D(v)|$ and $\deg_D^-(v) = |\mathcal{I}_D(v)|$ for every $v \in \mathcal{V}(G)$. Then we have

$$|\mathcal{I}_D(v)| = \deg_G(v) - |\mathcal{O}_D(v)| \geq \deg_G(v) - \left\lceil \frac{\deg_G(v)}{2} \right\rceil = \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor, \ \forall v \in \mathcal{V}(G). \tag{5}$$

On the other hand, the main condition $|f(v)| > \left\lceil \frac{\deg_G(v)}{2} \right\rceil$, $\forall v \in \mathcal{V}(G)$, of Theorem 3.2 yields

$$|[0 : \deg_G(v)] \setminus f(v)| = \deg_G(v) + 1 - |f(v)| < \deg_G(v) + 1 - \left\lceil \frac{\deg_G(v)}{2} \right\rceil,$$

thereby we arrive at

$$|[0 : \deg_G(v) \setminus f(v)]| \leq \deg_G(v) - \left\lceil \frac{\deg_G(v)}{2} \right\rceil = \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor, \ \forall v \in \mathcal{V}(G). \tag{6}$$

So combining two bounds (5) and (6) together gives $|[0 : \deg_G(v) \setminus f(v)]| \leq |\mathcal{I}_D(v)|$ for all $v \in \mathcal{V}(G)$. Hence for every $v \in \mathcal{V}(G)$, it is possible to choose a subset $\mathcal{F}(v)$ of $\mathcal{I}_D(v)$ such that $|\mathcal{F}(v)| = |[0 : \deg_G(v) \setminus f(v)]|$. Then one can see that for any two distinct vertices $u, v \in \mathcal{V}(G)$, we have

$$\mathcal{F}(u) \cap \mathcal{F}(v) = \varnothing. \tag{7}$$

As the final step, we let $t_e := |\{v \in \mathcal{V}(G) : e \in \mathcal{F}(v)\}|$ for each $e \in \mathcal{E}(G)$. Due to the pairwise disjoint property (7) of the collection $\{\mathcal{F}(v) : v \in \mathcal{V}(G)\}$, we obtain $t_e \in \{0, 1\}$ for every $e \in \mathcal{E}(G)$. Furthermore, the following holds:

$$\begin{aligned}
\sum_{e \in \mathcal{E}(G)} t_e &= \sum_{e \in \mathcal{E}(G)} |\{v \in \mathcal{V}(G) : e \in \mathcal{F}(v)\}| \\
&= \sum_{v \in \mathcal{V}(G)} |\{e \in \mathcal{E}(G) : e \in \mathcal{F}(v)\}| \\
&= \sum_{v \in \mathcal{V}(G)} |\mathcal{F}(v)| \\
&= \sum_{v \in \mathcal{V}(G)} |[0 : \deg_G(v) \setminus f(v)]| \\
&= \deg(g).
\end{aligned} \tag{8}$$

Therefore, $\prod_{e \in \mathcal{E}(G)} x_e^{t_e}$ is the desired monomial with non-zero coefficient satisfying

$$t_e \in \{0, 1\}\,, \ \forall e \in \mathcal{E}(G), \quad \text{and} \quad \sum_{e \in \mathcal{E}(G)} t_e = \deg(g) = \sum_{v \in \mathcal{V}(G)} |[0 : \deg_G(v)] \setminus f(v)|\,.$$

So,

$$\prod_{e\in\mathcal{E}(G)} x_e^{t_e} = \prod_{v\in\mathcal{V}(G)} \left(\prod_{e\in\mathcal{F}(v)} x_e\right)$$

is a monomial of degree $\deg(g)$. Here, one can conclude that the coefficient of the monomial $\prod_{e\in\mathcal{E}(G)} x_e^{t_e}$ in the expansion of $g(\mathbf{x})$ is positive:

$$\begin{aligned}
\left(\text{the coefficient of } \prod_{e\in\mathcal{E}(G)} x_e^{t_e} \text{ in } g(\mathbf{x})\right) &= \left(\text{the coefficient of } \prod_{v\in\mathcal{V}(G)} \left(\prod_{e\in\mathcal{F}(v)} x_e\right) \text{ in } \prod_{v\in\mathcal{V}(G)} \left(\sum_{\substack{e\in\mathcal{E}(G):\\ v\in e}} x_e\right)^{|\mathcal{F}(v)|}\right) \\
&\geq \prod_{v\in\mathcal{V}(G)} \left(\text{the coefficient of } \prod_{e\in\mathcal{F}(v)} x_e \text{ in } \left(\sum_{\substack{e\in\mathcal{E}(G):\\ v\in e}} x_e\right)^{|\mathcal{F}(v)|}\right) \\
&= \prod_{v\in\mathcal{V}(G)} \frac{\left(\sum_{e\in\mathcal{F}(v)} 1\right)!}{\prod_{e\in\mathcal{F}(v)} 1!} \\
&= \prod_{v\in\mathcal{V}(G)} |\mathcal{F}(v)|! > 0
\end{aligned}$$

Hence, the combinatorial nullstellensatz implies that there exists a vector $\mathbf{x}^* = (x_e^* : e \in \mathcal{E}(G)) \in \{0,1\}^{\mathcal{E}(G)}$ such that $g(\mathbf{x}^*) \neq 0$. Let $H$ be a spanning subgraph of $G$ with $\mathcal{E}(H) := \{e \in \mathcal{E}(G) : x_e^* = 1\}$. Then it is easy to see that $H$ becomes a generalized $f$-factor in $G$, and this completes the proof of Theorem 3.2.

□

*Remark* 3.1. Theorem 3.2 is a tight result in the following sense: given a finite simple, connected graph $G$ together with a function $f : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ with $f(v) \subseteq [0 : \deg_G(v)]$ for every $v \in \mathcal{V}(G)$, if $|f(v)| \leq \left\lceil \frac{\deg_G(v)}{2} \right\rceil$ for some $v \in \mathcal{V}(G)$, then $G$ may not have a generalized $f$-factor. As a simple counterexample, we consider the complete bipartite graph $K_{2n,2n}$ and a function $f : \mathcal{V}(K_{2n,2n}) := A \cup B \to 2^{\mathbb{Z}}$ defined by

$$f(v) := \begin{cases} [0:n] = \{0,1,\cdots,n\} & \text{if } v \in A; \\ [n:2n] = \{n,n+1,\cdots,2n\} & \text{if } v \in B \setminus \{v^*\}; \\ [n+1:2n] = \{n+1,n+2,\cdots,2n\} & \text{if } v = v^*, \end{cases}$$

where $v^* \in B$ is an arbitrarily chosen vertex in $B$. Note that the condition (1) of Theorem 3.2 only fails for the specified vertex $v^* \in B$. Assume on the contrary that $K_{2n,2n}$ has a generalized $f$-factor, say $H$. Since $H$ is a spanning subgraph of $K_{2n,2n}$ which is bipartite, we have

$$|\mathcal{E}(H)| = \sum_{v\in A} \deg_H(v) = \sum_{v\in B} \deg_H(v). \tag{9}$$

On the other hand, we see that

$$\begin{aligned}
&\sum_{v\in A} \deg_H(v) \geq n \cdot |A| = 2n^2; \\
&\sum_{v\in B} \deg_H(v) = \sum_{v\in B\setminus\{v^*\}} \deg_H(v) + \deg_H(v^*) \geq n \cdot |B \setminus \{v^*\}| + (n+1) = 2n^2 + 1,
\end{aligned}$$

and this fact violates the equation (9). Hence, $K_{2n,2n}$ has no generalized $f$-factors.

The following theorem gives a sufficient condition for the existence of partial generalized $f$-factors in a finite simple graph $G$:

**Theorem 3.3** ([12])**.** *Let $G$ be a finite simple graph, and $f : \mathcal{V}(G) \to 2^{\mathbb{Z}}$ be a function with $f(v) \subseteq [0 : \deg_G(v)]$ for every $v \in \mathcal{V}(G)$. If we have*

$$|\mathcal{E}(G)| > \sum_{v \in \mathcal{V}(G)} |[\deg_G(v)] \setminus f(v)|, \tag{10}$$

*where $[\deg_G(v)] := \{1, 2, \cdots, \deg_G(v)\}$ for $v \in \mathcal{V}(G)$, then $G$ contains a non-trivial partial $f$-factor, i.e., a partial $f$-factor whose edge set is non-empty.*

*Proof of Theorem 3.3.*

We consider the polynomial $h(\mathbf{x})$ over $\mathbb{R}$ with $|\mathcal{E}(G)|$ variables $x_e$, $e \in \mathcal{E}(G)$, defined by

$$h(\mathbf{x}) := \underbrace{\prod_{v \in \mathcal{V}(G)} \left[ \prod_{c \in [\deg_G(v)] \setminus f(v)} \left\{ \frac{c - \left( \sum_{e \in \mathcal{E}(G)} \mathbb{I}_G(v, e) \cdot x_e \right)}{c} \right\} \right]}_{=:\ h_1(\mathbf{x})} - \underbrace{\prod_{e \in \mathcal{E}(G)} (1 - x_e)}_{=:\ h_2(\mathbf{x})}, \ \forall \mathbf{x} = (x_e : e \in \mathcal{E}(G)) \in \mathbb{R}^{\mathcal{E}(G)},$$

where $\mathbb{I}_G(\cdot, \cdot) : \mathcal{V}(G) \times \mathcal{E}(G) \to \{0, 1\}$ refers to the incidence matrix of $G$. Then, $g(\mathbf{0}) = 1 - 1 = 0$, where $\mathbf{0} \in \mathbb{R}^{\mathcal{E}(G)}$ is the zero vector. Also from the condition (10), we see that

$$\deg(h_1) = \sum_{v \in \mathcal{V}(G)} |[\deg_G(v)] \setminus f(v)| < |\mathcal{E}(G)| = \deg(h_2),$$

and this implies $\deg(h) = \max \{\deg(h_1), \deg(h_2)\} = |\mathcal{E}(G)|$. So the largest degree monomial in the expansion of $h(\mathbf{x})$ is precisely $(-1)^{|\mathcal{E}(G)|-1} \prod_{e \in \mathcal{E}(G)} x_e$ and thus we can apply the combinatorial nullstellensatz: there is a vector $\mathbf{x}^* = (x_e^* : e \in \mathcal{E}(G)) \in \{0, 1\}^{\mathcal{E}(G)}$ such that $h(\mathbf{x}^*) \neq 0$. Since $h(\mathbf{0}) = 0$, we know that the vector $\mathbf{x}^*$ is non-zero and thus $h_2(\mathbf{x}^*) = 0$. Therefore, we have

$$0 \neq h(\mathbf{x}^*) = h_1(\mathbf{x}^*).$$

This implies that $\sum_{e \in \mathcal{E}(G)} \mathbb{I}_G(v, e) \cdot x_e^* \in \tilde{f}(v) = f(v) \cup \{0\}$ for every $v \in \mathcal{V}(G)$. Let $H$ be a spanning subgraph of $G$, where $\mathcal{E}(H) := \{e \in \mathcal{E}(G) : x_e^* = 1\} \neq \varnothing$. Then, $\deg_H(v) \in \tilde{f}(v)$ for every $v \in \mathcal{V}(H) = \mathcal{V}(G)$ and therefore $H$ is a non-trivial partial generalized $f$-factor in $G$.

□

*Remark* 3.2. Note that Theorem 3.3 can be easily extended to the case of hypergraphs, and in fact exactly the same sufficient condition as described in Theorem 3.3 assures the existence of partial generalized $f$-factors in a given finite hypergraph $H$. Moreover, Theorem 3.3 is known to be best possible even for the case of hypergraphs. Let $H$ be any finite hypergraph such that there exists a vertex $v^* \in \mathcal{V}(H)$ with $v^* \in e$ for every $e \in \mathcal{E}(H)$, and $\{e \setminus \{v^*\} : e \in \mathcal{E}(H)\}$ is a collection of pairwise disjoint subsets of $\mathcal{V}(H)$. Then it's clear that

$$\deg_H(v) = \begin{cases} |\mathcal{E}(H)| & \text{if } v = v^*; \\ 1 & \text{otherwise.} \end{cases}$$

Define a function $f : \mathcal{V}(H) \to 2^{\mathbb{Z}}$ by

$$f(v) := \begin{cases} \{0\} & \text{if } v = v^*; \\ \{0,1\} & \text{otherwise.} \end{cases}$$

Then one can see that

$$|\mathcal{E}(H)| = \sum_{v \in \mathcal{V}(H)} |[\deg_H(v)] \setminus f(v)| \,.$$

We claim that $H$ has no non-trivial partial generalized $f$-factors. Assume on the contrary that there exists a non-trivial partial generalized $f$-factor $K$ in $H$. Then $\varnothing \neq \mathcal{E}(K) \subseteq \mathcal{E}(H)$ and thus $\deg_K(v^*) = |\mathcal{E}(K)| > 0$. This implies that $\deg_K(v^*) \notin f(v^*) \cup \{0\} = \{0\}$, which contradicts the assumption that $K$ is a non-trivial partial generalized $f$-factor in $H$.

## 4 A Slight Extension

Throughout this section, we will introduce a brief idea which may play a role as a useful extension of Theorem 3.2 to the case of hypergraphs. To begin with, we extend our key notions to the case of hypergraphs.

**Definition 4.1** (The generalization of Definition 1.3)**.** Let $H$ be a finite hypergraph and $f : \mathcal{V}(H) \to 2^{\mathbb{Z}}$ be a function such that $f(v) \subseteq [0 : \deg_H(v)]$ for every $v \in \mathcal{V}(H)$. A *generalized $f$-factor in $H$* is a spanning sub-hypergraph $K$ of $H$ such that $\deg_K(v) \in f(v)$ for every $v \in \mathcal{V}(K) = \mathcal{V}(H)$. A *partial generalized $f$-factor in $H$* refers to a generalized $\tilde{f}$-factor in $H$, where $\tilde{f}(v) : \mathcal{V}(H) \to 2^{\mathbb{Z}}$ is defined by $\tilde{f}(v) := f(v) \cup \{0\}$ for every $v \in \mathcal{V}(H)$.

We now introduce the notion of orientations in hypergraphs [6]. The following definition provides a fully generalized notion of orientations in undirected graphs.

**Definition 4.2** (Orientation in hypergraphs)**.** An *orientation on a hypergraph $H$* is a function $D : \mathcal{E}(H) \to \mathcal{V}(H)$ such that $D(e) \in e$ for every hyperedge $e \in \mathcal{E}(H)$, *i.e.*, a function which assigns a *head node* to every hyperedge in $\mathcal{E}(H)$. Also, we let

$$\mathcal{I}_D(v) := \{e \in \mathcal{E}(H) : v \in e \text{ and } D(e) = v\}\,;$$
$$\mathcal{O}_D(v) := \{e \in \mathcal{E}(H) : v \in e \text{ and } D(e) \neq v\}$$

for every $v \in \mathcal{V}(H)$.

By following the arguments in the proof of Theorem 3.2, one can prove the following sufficient condition for the existence of generalized $f$-factors in a given finite hypergraph:

**Theorem 4.1.** *Let $H$ be a finite hypergraph, i.e., a hypergraph with $|\mathcal{V}(H)| < \infty$, and $f : \mathcal{V}(H) \to 2^{\mathbb{Z}}$ be a function such that $f(v) \subseteq [0 : \deg_H(v)]$ for every $v \in \mathcal{V}(H)$. If there is an orientation $D : \mathcal{E}(H) \to \mathcal{V}(H)$ on $H$ satisfies one of the following properties:*

*(i) $|f(v)| \geq |\mathcal{O}_D(v)| + 1$ for every $v \in \mathcal{V}(H)$;*

*(ii) $|f(v)| \geq |\mathcal{I}_D(v)| + 1$ for every $v \in \mathcal{V}(H)$,*

*then $H$ has a generalized $f$-factor.*

*Proof of Theorem 4.1.*

Here, we only prove the presence of generalized $f$-factors in a given finite hypergraph $H$ for the case in which the property (i) holds, and the case when the property (ii) holds can be shown via exactly the same argument. We consider the polynomial $g(\mathbf{x})$ over $\mathbb{R}$ with $|\mathcal{E}(H)|$ variables $x_e$, $e \in \mathcal{E}(H)$, defined by

$$g(\mathbf{x}) := \prod_{v \in \mathcal{V}(H)} \left[ \prod_{c \in [0:\deg_H(v)] \setminus f(v)} \left\{ \left( \sum_{e \in \mathcal{E}(H)} \mathbb{I}_H(v, e) \cdot x_e \right) - c \right\} \right], \ \forall \mathbf{x} \in \mathbb{R}^{\mathcal{E}(H)},$$

where $\mathbb{I}_H(\cdot, \cdot) : \mathcal{V}(H) \times \mathcal{E}(H) \to \{0, 1\}$ refers to the *incidence matrix* of the hypergraph $H$. Then it's clear that

$$\deg(g) = \sum_{v \in \mathcal{V}(H)} \left\{ \deg_H(v) + 1 - |f(v)| \right\}. \tag{11}$$

In order to apply the combinatorial nullstellensatz, we show that there is a monomial term $a \cdot \prod_{e \in \mathcal{E}(H)} x_e^{t_e}$ in the expansion of $g(\mathbf{x})$, where $a > 0$ is a real coefficient and $t_e \in \{0, 1\}$ for each $e \in \mathcal{E}(H)$ together with

$$\sum_{e \in \mathcal{E}(H)} t_e = \deg(g) = \sum_{v \in \mathcal{V}(H)} \left\{ \deg_H(v) + 1 - |f(v)| \right\}.$$

Since the given orientation $D : \mathcal{E}(H) \to \mathcal{V}(H)$ satisfies the property

$$|\mathcal{I}_D(v)| = \deg_H(v) - |\mathcal{O}_D(v)| \geq \deg_H(v) + 1 - |f(v)|, \ \forall v \in \mathcal{V}(H),$$

it is possible to take $\mathcal{F}(v)$ as an arbitrary subset of $\mathcal{I}_D(v)$ so that $|\mathcal{F}(v)| = \deg_H(v) + 1 - |f(v)|$ for every $v \in \mathcal{V}(H)$. Then, one can see that $\{\mathcal{F}(v) : v \in \mathcal{V}(H)\}$ is a collection of pairwise disjoint subsets of $\mathcal{E}(H)$. Let $t_e := |\{v \in \mathcal{V}(H) : e \in \mathcal{F}(v)\}|$ for each $e \in \mathcal{E}(H)$. Thanks to the pairwise disjoint property of the collection $\{\mathcal{F}(v) : v \in \mathcal{V}(H)\}$, we have $t_e \in \{0, 1\}$ for every $e \in \mathcal{E}(H)$. Furthermore,

$$\begin{aligned}
\sum_{e \in \mathcal{E}(H)} t_e &= \sum_{e \in \mathcal{E}(H)} |\{v \in \mathcal{V}(H) : e \in \mathcal{F}(v)\}| \\
&= |\{(v, e) \in \mathcal{V}(H) \times \mathcal{E}(H) : e \in \mathcal{F}(v)\}| \\
&= \sum_{v \in \mathcal{V}(H)} |\mathcal{F}(v)| \\
&= \sum_{v \in \mathcal{V}(H)} \left\{ \deg_H(v) + 1 - |f(v)| \right\} \\
&\overset{\text{(a)}}{=} \deg(g),
\end{aligned}$$

where the step (a) follows from the equation (11). Thus

$$\prod_{e \in \mathcal{E}(H)} x_e^{t_e} = \prod_{v \in \mathcal{V}(H)} \left( \prod_{e \in \mathcal{F}(v)} x_e \right)$$

is the monomial contained in the expansion of $g(\mathbf{x})$ of the required degree $\deg(g)$. Here, one can see that

$$\begin{aligned}\left(\text{the coefficient of } \prod_{e\in\mathcal{E}(H)} x_e^{t_e} \text{ in } g(\mathbf{x})\right) &= \left(\text{the coefficient of } \prod_{v\in\mathcal{V}(H)}\left(\prod_{e\in\mathcal{F}(v)} x_e\right) \text{ in } \prod_{v\in\mathcal{V}(H)}\left(\sum_{\substack{e\in\mathcal{E}(H):\\ v\in e}} x_e\right)^{|\mathcal{F}(v)|}\right)\\ &\geq \prod_{v\in\mathcal{V}(H)}\left(\text{the coefficient of } \prod_{e\in\mathcal{F}(v)} x_e \text{ in } \left(\sum_{\substack{e\in\mathcal{E}(H):\\ v\in e}} x_e\right)^{|\mathcal{F}(v)|}\right)\\ &= \prod_{v\in\mathcal{V}(H)} \frac{\left(\sum_{e\in\mathcal{F}(v)} 1\right)!}{\prod_{e\in\mathcal{F}(v)} 1!}\\ &= \prod_{v\in\mathcal{V}(G)} |\mathcal{F}(v)|! > 0,\end{aligned}$$

thereby the coefficient of the monomial term $\prod_{e\in\mathcal{E}(H)} x_e^{t_e}$ in the expansion of $g(\mathbf{x})$ is positive. Hence, the combinatorial nullstellensatz implies that there is a vector $\mathbf{x}^* = (x_e : e \in \mathcal{E}(H)) \in \{0,1\}^{\mathcal{E}(H)}$ such that $g(\mathbf{x}^*) \neq 0$. Let $K$ be the spanning sub-hypergraph of $H$ with $\mathcal{E}(K) := \{e \in \mathcal{E}(H) : x_e^* = 1\}$. Then, it's easy to see that $K$ is a generalized $f$-factor in the hypergraph $H$ as desired.

$\square$

*Remark* 4.1. Let $H$ be a hypergraph and $D : \mathcal{E}(H) \to \mathcal{I}(H)$ be any orientation on $H$. For each $v \in \mathcal{V}(H)$, we may regard the size $|\mathcal{I}_D(v)|$ of the set of hyperedges in $H$ for which $v$ is their head node with respect to the orientation $D$ as the *in-degree* of $v$ under the orientation $D$, and the size $|\mathcal{O}_D(v)|$ of the set of hyperedges in $H$ for which $v$ is one of their tail nodes with respect to the orientation $D$ as the *out-degree* of $v$ under the orientation $D$. Indeed when $H$ is a simple graph, *i.e.*, a 2-uniform hypergraph, we have $|\mathcal{I}_D(v)| = \deg_D^-(v)$ and $|\mathcal{O}_D(v)| = \deg_D^+(v)$ for every $v \in \mathcal{V}(H)$, where $D : \mathcal{E}(H) \to \mathcal{V}(H)$ is any orientation on $H$.

In the viewpoint of Theorem 4.1, it is worthwhile investigating sufficient conditions for the existence of orientations on a given finite hypergraph so that the out-degrees of vertices (see Remark 4.1 for details) are *not too large*. We would like to emphasize that Lemma 3.1 investigates the existence of such *nice orientations* for the case of finite simple, connected graphs, and one can obtain the main result, Theorem 3.2 in [12], as a corollary of our extension Theorem 4.1 together with Lemma 3.1. See [6, 8] for further studies of the existence of orientations on finite hypergraphs. Finally, we conclude this section by remarking that exactly the same conclusion as Theorem 3.3 also holds for the case of finite hypergraphs and its proof is completely the same with the proof of Theorem 3.3.

## 5 Concluding Remarks and Future Directions

We recall that the following result is one of the homework problems (*Problem 4.2* in Homework #4) given in the graduate course on combinatorics (MAS575) in the spring semester of 2021:

**Theorem 5.1** (*Theorem 6.1* in [3]). *Let $G$ be a finite simple undirected graph, and $p$ be a prime number. If the average degree of $G$ is greater than $2p-2$, and the maximum degree of $G$ is at most $2p-1$, then $G$ contains a $p$-regular subgraph.*

In the paper [4], the authors proved that Theorem 5.1 holds when $p$ is any prime power via another combinatorial argument. From these results, we can consider their natural extensions as follows:

**Definition 5.1** ($f$-factors mod $p$ in $G$ [12])**.** Let $G$ be a finite simple undirected graph, $p$ be a prime power, and $f : \mathcal{V}(G) \to \mathbb{Z}$. An *$f$-factor mod $p$ in $G$* is a spanning subgraph $H$ of $G$ such that $\deg_H(v) \equiv f(v) \pmod{p}$ for all $v \in \mathcal{V}(H) = \mathcal{V}(G)$. Also, a *partial $f$-factor mod $p$ in $G$* refers to a subgraph $L$ of $G$ (not necessarily be a spanning subgraph of $G$) such that $\deg_L(v) \equiv f(v) \pmod{p}$ for all $v \in \mathcal{V}(L)$.

In the viewpoint of Theorem 5.1, one can ask the following question:

**Question 1.** *Given any prime power $p$, does there exist a partial $f$-factor mod $p$ in a finite simple undirected graph $G$ whose average degree is sufficiently large (but constant)?*

This question still remains open, and even the case for which $f(v) = q$ for all $v \in \mathcal{V}(G)$ has been considered as a challenging task. Also, we can ask the following intriguing question:

**Question 2.** *Given any prime power $p$, what can be a sufficient condition for the existence of a $f$-factor mod $p$ in a finite simple undirected graph $G$?*

Regarding Question 2, when $p$ is a prime number, we see that an $f$-factor exists in $G$ in $G$ if and only if the polynomial $p(\mathbf{x})$ over $\mathbb{F}_p$ with $|\mathcal{E}(G)|$ variables $x_e$, $e \in \mathcal{E}(G)$, defined by

$$p(\mathbf{x}) := \prod_{v \in \mathcal{V}(G)} \left[ 1 - \left\{ \left( \sum_{e \in \mathcal{E}(G)} \mathbb{I}_G(v, e) \cdot x_e \right) - f(v) \right\}^{p-1} \right], \ \forall \mathbf{x} = (x_e : e \in \mathcal{E}(G)) \in \mathbb{F}_p^{\mathcal{E}(G)},$$

is non-zero for some instance $\mathbf{x} \in \{0, 1\}^{\mathcal{E}(G)}$ of the binary variables due to the Fermat's little theorem. From this observation, one can partially answer to Question 2 by using the combinatorial nullstellensatz for some particular sceanrios. One of such scenarios can be characterized as follows:

**Theorem 5.2.** *Let $p$ be a prime number, and $G$ be a finite simple undirected graph with the average degree $2p - 2$. If the number of orientations on $G$ with the out-degree of each vertex $p - 1$ is not divisible by $p$, then $G$ has an $f$-factor mod $p$ for any function $f : \mathcal{V}(G) \to \mathbb{Z}$.*

*Proof of Theorem 5.2.*

Let $N$ denote the number of orientations on $G$ with the out-degree of each vertex $p - 1$. From

$$\frac{2\,|\mathcal{E}(G)|}{|\mathcal{V}(G)|} = \frac{1}{|\mathcal{V}(G)|} \sum_{v \in \mathcal{V}(G)} \deg_G(v) = 2(p-1),$$

we have $\deg(p) = (p-1)\,|\mathcal{V}(G)| = |\mathcal{E}(G)|$. Thus, $\prod_{e \in \mathcal{E}(G)} x_e$ is the monomial term contained in the expansion of $p(\mathbf{x})$ of the required degree $\deg(p) = |\mathcal{E}(G)|$. We claim that the coefficient of the monomial $\prod_{e \in \mathcal{E}(G)} x_e$ in the expansion of $p(\mathbf{x})$ is not divisible by $p$, *i.e.*, non-zero in the field $\mathbb{F}_p$. Since

$$\begin{aligned}
\left( \text{the coefficient of } \prod_{e \in \mathcal{E}(G)} x_e \text{ in } p(\mathbf{x}) \right) &= (-1)^{|\mathcal{V}(G)|} \left( \text{the coefficient of } \prod_{e \in \mathcal{E}(G)} x_e \text{ in } \prod_{v \in \mathcal{V}(G)} \left( \sum_{\substack{e \in \mathcal{E}(G): \\ v \in e}} x_e \right)^{p-1} \right) \\
&= (-1)^{|\mathcal{V}(G)|} (p-1)! \cdot N \\
&\not\equiv 0 \pmod{p},
\end{aligned}$$

our claim follows. So by the combinatorial nullstellensatz, there is a vector $\mathbf{x}^* = (x_e^* : e \in \mathcal{E}(G)) \in \{0,1\}^{\mathcal{E}(G)}$ such that $g(\mathbf{x}^*) \not\equiv 0 \,(\mathrm{mod}\ p)$. Let $H$ be the spanning subgraph of $G$ with $\mathcal{E}(H) := \{e \in \mathcal{E}(G) : x_e^* = 1\}$. Then, it's clear that $H$ is an $f$-factor mod $p$ in $G$.

□

From Theorem 5.2, we can derive the following two corollaries.

**Corollary 5.1.** *Let $p$ be a prime number, and $G$ be a finite simple undirected $(2p-2)$-regular graph. If the number of Eulerian orientations on $G$ is not divisible by $p$, then $G$ has an $f$-factor mod $p$ for any function $f : \mathcal{V}(G) \to \mathbb{Z}$.*

**Corollary 5.2.** *For any prime number $p$, the number of Eulerian orientations on a bipartite $(2p-2)$-regular graph is divisible by $p$.*

*Proof of Corollary 5.2.*

By considering the contraposition of Corollary 5.1, it suffices to find an example of $f : \mathcal{V}(G) \to \mathbb{Z}$ for any bipartite $(2p-2)$-regular graph $G$ so that $G$ has no $f$-factors mod $p$. Let $G$ be any bipartite $(2p-2)$-regular graph with bipartition $\mathcal{V}(G) = \mathcal{A}(G) \cup \mathcal{B}(G)$. Define $f : \mathcal{V}(G) \to \mathbb{Z}$ by

$$f(v) := \begin{cases} 0 & \text{if } v \in \mathcal{A}(G); \\ 1 & \text{if } v \in \mathcal{B}(G). \end{cases}$$

Assume on the contrary that $G$ has an $f$-factor mod $p$, say $H$. For any $v \in \mathcal{B}(G)$, we know that $\deg_H(v) = f(v) = 1$. Thus there exists a unique vertex $u \in \mathcal{A}(G)$ such that $\{u, v\} \in \mathcal{E}(H)$. Then we obtain

$$1 \leq \deg_H(u) = f(u) = 0,$$

and this gives us a contradiction! Hence, $G$ has no $f$-factors mod $p$. So from the contraposition of Corollary 5.1, we can conclude that the number of Eulerian orientations on $G$ is divisible by $p$.

□

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